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We show that the generalized Radon transform arising in a model of pure industry taking into account the substitution of production factors at the micro-level is an integral Fourier operator satisfying the condition of microlocal regularity. We describe a method for reconstruction of singularities of a function from the singularities of its generalized Radon transform.

Keywords: generalized Radon transform, microlocal analysis, integral Fourier operators, model of pure industry, constant elasticity of substitution

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1 Introduction

Modern production systems are experiencing such effects as standartization and globalization which, in particular, lead to the situation when different production factors (resources) can substitute each other in the production process. One of the actual problems of the production theory is to give a mathematical description of the production process in a pure industry taking into account this effect of substitution of production factors and allowing its quantitative analysis.

One of the models of pure industry taking into account the aforementioned effect was proposed by A. A. Shananin in [Sha2], [Sha3]. This model generalizes the Houthakker–Johansen model of pure industry (see [Sha1]) allowing an arbitrary continuous positively homogeneous monotone function as the production function at the micro-level whereas the Houthakker–Johansen model allowed only the Leontief production functions.

In the present paper we consider Shananin's model of pure industry in one of the most important particular cases, when the elasticity of substitution of production factors is constant. We also restrict our attention to the case of

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two production factors in order to simplify the mathematical formulas but the multidimensional case is completely similar.

In Shananin's model the main tool of quantitative description of a pure industry at the macro-level is the profit function which, under the assumption of constant elasticity of substitution of production factors, is given by the following formula:

$$\Pi_\alpha(p_0, p_1, p_2) = \int_0^{p_0} \int_0^t \frac{1}{\tau} R_\alpha\left(\frac{p_1}{\tau}, \frac{p_2}{\tau}\right) d\tau\, dt \quad \alpha \le 1, \tag{1.1}$$

where p_0 is the unit price of the final product, p_0, p_1 are the unit prices of the production factors, $(1-\alpha)$ is the elasticity of substitution of production factors and R_α is the generalized Radon transform on the positive quadrant $X = \{(x_1, x_2)\colon x_1 > 0, x_2 > 0\}$ which maps a sufficiently regular real-valued function u on X to the function $R_\alpha u$ on $P = \{(p_1, p_2)\colon p_1 > 0, p_2 > 0\}$ defined as follows:

$$R_\alpha u(p) = \int_{X_{\alpha,p}} u(x)\, \omega_{\alpha,p}, \tag{1.2}$$

where $p = (p_1, p_2)$,

$$X_{\alpha,p} = \big\{(x_1, x_2) \in X\colon q_\alpha(p_1 x_1, p_2 x_2) = 1\big\}, \tag{1.3}$$

$$q_\alpha(x_1, x_2) = (x_1^\alpha + x_2^\alpha)^{1/\alpha}, \quad x_1, x_2 > 0, \tag{1.4}$$

$\omega_{\alpha,x}$ is the Gelfand–Leray 1-form given by the interior product:

$$\omega_{\alpha,p} = d_x q_\alpha(p_1 x_1, p_2 x_2) \lrcorner (dx_1 \wedge dx_2), \tag{1.5}$$

and orientation on $X_{\alpha,p}$ is given by the volume form $|\nabla_x q_\alpha(p_1 x_1, p_2 x_2)| \omega_{\alpha,p}$.

Recall that by definition $\omega_{\alpha,p}$ is the restriction to $X_{\alpha,p}$ of any 1-form $\widetilde{\omega}_{\alpha,p}$ on X satisfying

$$d_x q_\alpha(p_1 x_1, p_2 x_2) \wedge \widetilde{\omega}_{\alpha,p} = dx_1 \wedge dx_2.$$

The form $\omega_{\alpha,p}$ doesn't depend on the choice of $\widetilde{\omega}_{\alpha,p}$.

In the setting of Shananin's model q_α is the unit cost function of the final product, u has the meaning of density of distribution of production powers over technologies and $\Pi_\alpha(p_0, p_1, p_2)$ is the total profit of the industry for a production cycle. In this context function q_α defined in (1.4) is called the constant elasticity of substitution function (CES-function), corresponding to elasticity $(1-\alpha)$.

The case $\alpha = 1$ corresponds to production systems with no substitution of production factors at the microlevel. This assumption is not fulfilled in the majority of production systems experiencing the effects of standartization and globalization. In this case transform (1.2) is the classical Radon transform in $\mathbb{R}^2$ with limited data: only those straight lines that intersect both coordinate axes in the first (nonnegative) quadrant are considered. The properties of this transform were studied in [HS].

Passing $\alpha \to 0$ in (1.2) we obtain the transform R_0 that integrates function u over the sets $x_1 x_2 = \text{const}$. This case corresponds to production systems with the Cobb–Douglas production functions. Note that transform R_0 maps a function of two variables to a function of one variable (after an appropriate change of variables). We exclude this case from consideration.

As it was mentioned before, the profit function Π_α is the main tool of quantitative description of the production system at the macro-level (as well as the production function, but they are dual, see [Sha2]). Note that we consider Π_α (resp. R_α) as an integral transform acting on functions u defined in X. From this point of view it follows from formula (1.1) that transforms Π_α and R_α are essentially the same (they are related by differentiation and integration, respectively). This means that the problem of quantitative analysis of production process in Shananin's model reduces to the study of the transform R_α.

We are mainly interested in the following problems for transform R_α.

Problem 1 (Uniqueness). *Find sufficient conditions for injectivity of transform R_α on a suitable space. Find examples of non-injectivity of transform R_α.*

Problem 2 (Characterization). *Decribe the range of transform R_α on a suitable space.*

Problem 3 (Inversion). *Find a stable algorithm (or formula) for inversion of R_α on a suitable space. Obtain the stability estimates.*

Problem 4 (Microlocal properties). *Describe the propagation of singularities by transform R_α.*

By «suitable spaces» we understand the spaces which naturally arise in the above setting of the generalized model of pure industry, for example, the space of non-negative compactly supported continuous functions or more general spaces.

Transform R_α was introduced in Ref. [HS] in the case when $\alpha = 1$. It was studied for the first time under the assumption that $\alpha > 0$ in Ref. [Ag1]. In Ref. [Ag2] the more general transform was considered, where integration in (1.2) is over the level hypersurfaces of an arbitrary positively homogeneous function in positive orthant satisfying some regularity assumptions.

Problem 1 was considered in Ref. [Ag2]. In this work it was shown that transform R_α is injective on the subspace of non-negative functions (and even measures) growing not faster than exponents at infinity. Besides, it was shown that the ranges of transforms R_α and R_β for $\alpha > 0$, $\beta > 0$, $\alpha \neq \beta$, on a subspace of functions (and even measures) with sufficient decay at infinity intersect only by zero function.

Problem 2 was studied in Ref. [Ag1]. In a subsequent work we will present results on Problem 2 in the case of more general integral transform over the level hypersurfaces of an arbitrary sufficiently regular positively homogeneous function.

Problem 3 was addressed in Ref. [Ag1] in the case of transform R_α and in Ref. [Ag2] in the case of more general integral transform.

Besides, Problems 1, 2, 3 for transform R_α were studied in Ref. [HS] in the case when $\alpha = 1$.

In the present paper we study Problem 4. More precisely, we show that R_α, $\alpha \in \mathbb{R} \setminus \{0,1\}$, is a Fourier integral operator satisfying the property of microlocal regularity and we propose a method for reconstruction of singularities, see Theorem 2.1 and Proposition 2.1 of Section 2.

We don't consider the case $\alpha = 1$ for the following reason. On the one hand, our method of proof of microlocal regularity is based on a version of method of stationary phase presented in Section 4 which uses the fact that the linear function $S(x,\xi) = \xi_1 x_1 + \xi_2 x_2$ at fixed ξ_1, ξ_2 with $\xi_1\xi_2 > 0$ attains the unique extremum $x^*(p,\xi)$ on $X_{\alpha,p}$ for any $p \in P$ when $\alpha \in \mathbb{R} \setminus \{0,1\}$ and this is not true when $\alpha = 1$. On the other hand, restrictions in space prevent us from studying the microlocal properties of R_1 as an exceptional case in the present paper. Finally, our study of transform R_α is mainly motivated by the study of production systems with substitution of production factors at the microlevel and this case corresponds to $\alpha \neq 1$.

It is important to note that in many applications related with generalized Radon transforms we are interested in anomalies in distribution of a quantity described by function u rather than in exact values of u. In this case the property of microlocal regularity allows us to estimate these anomalies via the singularities of $R_\alpha u$ without constructing the inverse operator R_α^{-1}.

Besides, the knowledge of microlocal properties of operator R_α is very important if we are interested in reconstruction problem for this operator from incomplete data (or in presence of noise). In particular, even if the reconstruction is possible it may be not convenient for numerical computations if the generalized Radon transform has bad microlocal properties.

On the other hand, even if the generalized Radon transform is not injective, it is sometimes possible to construct a convenient pseudo-inverse operator if the generalized Radon transform has good microlocal properties. For related discussion, see, e.g., Ref. [Fa].

In addition, the property of microlocal regularity (in the case of analytic singularities) was used by some authors to estimate the support of function u given the support of its generalized Radon transform, see, e.g., Refs. [BQ1], [BQ2], [Bo], [Qu2], [Qu3], [Kr].

Note that the history of microlocal approach to generalized Radon transforms goes back to Refs. [Gu], [GS].

Concerning the results given in literature on microlocal properties of generalized Radon transforms, see, e.g., Refs. [ABKQ], [Bo], [BQ1], [BQ2], [FLU], [FQ], [GU], [KLQ], [Qu2], [Qu3], [SU] and references therein.

2 Main results

To formulate main results we need to introduce some notations. For fixed $\alpha \in \mathbb{R} \setminus 0$, $x \in X$ we define

$$P_{\alpha,x} = \big\{(p_1,p_2) \in P \colon q_\alpha(p_1x_1, p_2x_2) = 1\big\},$$

with orientation given by the volume form $|\nabla_p q_\alpha(p_1x_1,p_2x_2)|\omega_{\alpha,x}$, where $\omega_{\alpha,x} = d_p q(p_1x_1,p_2x_2) \lrcorner (dp_1\wedge dp_2)$.

The dual transform R^*_α to R_α maps a sufficiently regular real-valued function v on P to the function $R^*_\alpha v$ on X given by the formula

$$R^*_\alpha v(x) = \int_{P_{\alpha,x}} v(p)\,\omega_{\alpha,x}. \tag{2.1}$$

Note that since $P\cong X$, in fact, transform R_α is self-dual.

Let Z_α be the following incidence relation between the points in X and P:

$$Z_\alpha \stackrel{\text{def}}{=\!=} \bigl\{(p_1,p_2;x_1,x_2)\in P\times X\colon q_\alpha(p_1x_1,p_2x_2)=1\bigr\}.$$

We consider Z_α as a submanifold of $P\times X$.

The total space of conormal bundle N^*Z_α of Z_α in $P\times X$ is given by the following formula:

$$\begin{gathered} N^*Z_\alpha = \bigl\{\bigl(p_1,p_2;\lambda\sum\nolimits_{j=1}^{2} x_j^\alpha p_j^{\alpha-1}dp_j;x_1,x_2;\lambda\sum\nolimits_{j=1}^{2} p_j^\alpha x_j^{\alpha-1}dx_j\bigr)\in \\ \in T^*P\times T^*X\cong T^*(P\times X)\colon q_\alpha(p_1x_1,p_2x_2)=1,\ \lambda\in\mathbb{R}\bigr\}. \end{gathered} \tag{2.2}$$

We denote by $\dot N^*Z_\alpha$ the manifold N^*Z_α with zero section removed.

In a similar way, the total space of conormal bundle $N^*X_{\alpha,p}$ of $X_{\alpha,p}$ in X is given by formula

$$\begin{gathered} N^*X_{\alpha,p} = \bigl\{(x_1,x_2;\lambda\sum\nolimits_{j=1}^{2} p_j^\alpha x_j^{\alpha-1}dx_j)\colon \\ (x_1,x_2)\in X,\ q_\alpha(p_1x_1,p_2x_2)=1,\ \lambda\in\mathbb{R}\bigr\},\quad p\in P. \end{gathered} \tag{2.3}$$

We define the canonical relation C_α to be the total space of the twisted conormal bundle of Z_α in $P\times X$ with zero section removed:

$$\begin{gathered} C_\alpha \stackrel{\text{def}}{=\!=} (\dot N^*Z_\alpha)' = \{(p,\eta dp;x,-\xi dx)\in T^*P\times T^*X\colon \\ (p,\eta dp,x,\xi dx)\in \dot N^*Z_\alpha\}, \end{gathered} \tag{2.4}$$

where

$$\begin{gathered} p=(p_1,p_2),\ \eta=(\eta_1,\eta_2),\ x=(x_1,x_2),\ \xi=(\xi_1,\xi_2), \\ \eta dp = \eta_1dp_1+\eta_2dp_2,\quad \xi dx=\xi_1dx_1+\xi_2dx_2. \end{gathered}$$

Theorem 2.1. *Let $\alpha\in\mathbb{R}\setminus\{0,1\}$ be fixed. Then:*

1. *R_α is a Fourier integral operator associated with canonical relation C_α. Hence R_α is the linear continuous operator from $C^\infty_c(X)$ to $C^\infty(P)$ and from $\mathscr{E}'(X)$ to $\mathscr{D}'(P)$, where $\mathscr{E}'(X)$ denotes the space of compactly supported distributions from $\mathscr{D}'(X)$;*

2. *(microlocal regularity) for any fixed $u\in\mathscr{E}'(X)$ the condition $p\notin\operatorname{sing\,supp} R_\alpha u$ is satisfied if and only if*

$$WF(u)\cap N^*X_{\alpha,p}=\varnothing, \tag{2.5}$$

where WF denotes the wave front set and $N^*X_{\alpha,p}$ is defined in (2.3);

3. for any $u \in \mathscr{E}'(X)$ if $(x, \xi dx) \in WF(R_\alpha u)$, $\xi = (\xi_1, \xi_2)$, then $\xi_1 \xi_2 > 0$.

Statement 2 of Theorem 2.1 tells us that if distribution $u \in \mathscr{E}'(X)$ has a singularity at point $x \in X_{\alpha,p}$ which is conormal to curve $X_{\alpha,p}$, then distribution $R_\alpha u$ has a singularity at point p. Furthermore, all singularities of $R_\alpha u$ arise in this way.

Note that since the dual transform R_α^* defined in (2.1) is given by the same formula as transform R_α (after changing x to p and vice versa), Theorem 2.1 also holds for R_α^* if we replace X by P, $X_{\alpha,p}$ by $P_{\alpha,x}$ and vice versa in its statement.

Theorem 2.1 is proved in Section 3. The proof of the "only if" part of statement 2 of Theorem 2.1 is based on Lemma 3.1 of Section 3 which is proved in Section 5.

In order to formulate the following result we need to introduce two notations. For fixed $0 < \delta < \frac{1}{2}$ chose any $\chi_\delta \in C^\infty(0, +\infty)$ such that

$$\chi_\delta(t) = \begin{cases} 0, & \text{if } t < \delta \text{ or } t > \delta^{-1}, \\ 1, & \text{if } t \geq 2\delta \text{ and } t \leq \frac{1}{2}\delta^{-1}. \end{cases} \tag{2.6}$$

For fixed $\alpha \in \mathbb{R} \setminus \{0, 1\}$ define $\gamma(\alpha)$ by formula

$$\gamma(\alpha) = \begin{cases} 1 + \alpha^{-1}, & \alpha > 1, \\ -1 + 3\alpha^{-1}, & 0 < \alpha < 1, \\ 1 - 3\alpha^{-1}, & \alpha < 0. \end{cases} \tag{2.7}$$

Proposition 2.1 (reconstruction of singularities). *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and $0 < \varepsilon < \frac{1}{2}$ be fixed. Define $\gamma(\alpha)$ by formula (2.7) and put $\delta = 2^{-\frac{1}{\alpha} - 1} \varepsilon^{\gamma(\alpha)}$. Then for any $u \in \mathscr{E}'(X)$ such that*

$$\operatorname{supp} u \subseteq \{(x_1, x_2) \in X : \varepsilon < x_j < \varepsilon^{-1},\ j = 1, 2\}, \tag{2.8}$$

and for all $(x, \xi dx) \in T^*X$, $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, satisfying $\varepsilon < x_j < \varepsilon^{-1}$, $j = 1$, 2, $\varepsilon < \frac{\xi_1}{\xi_2} < \varepsilon^{-1}$, the following formula holds:

$$(x, \xi dx) \in WF(u) \ \textit{ if and only if } \ (x, \xi dx) \in WF\big(R_\alpha^* \chi_\delta^1 \chi_\delta^2 R_\alpha u\big), \tag{2.9}$$

where χ_δ^j denotes the operator of multiplication by $\chi_\delta(p_j)$, $j = 1$, 2, and χ_δ is defined in formula (2.6).

Proposition 2.1 is proved in Section 3. The proof is based on Lemma 3.1 of Section 3.

Proposition 2.1 allows us to reconstruct singularities of u given $R_\alpha u$ and *a priori* information about bounds on support of u, by choosing an appropriate δ and applying operator $R_\alpha^* \chi_\delta^1 \chi_\delta^2$ to $R_\alpha u$. For numerical examples, see Section 6.

Remark 2.1. Note that replacing function q_α in formula (1.4) by the function of the same form but with $n \geq 3$ arguments, we could consider the generalized Radon transform over the level hypersurfaces of CES-function in dimension n.

The generalization of Theorem 2.1 and Proposition 2.1 to this case is straightforward. We consider the two-dimensional case in order to simplify the proofs.

Remark 2.2. In fact, statements 1, 3 and the "if" part of statement 2 of Theorem 2.1 remain valid if we replace function q_α in definition (1.2) by much more general smooth function q. In particular, we could consider functions $q \in C^\infty(X)$ satisfying the following conditions:

Q1. $q(\lambda x) = \lambda q(x)$, $x \in X$, $\lambda > 0$,

Q2. $q(x)$, $\partial_1 q(x)$, $\partial_2 q(x)$ are positive for $x \in X$, where $\partial_j q = \partial q/\partial x_j$, $j = 1, 2$;

Q3. for any fixed $p \in P$ the map $\Phi_{q,p}\colon X_{q,p} \to (0, +\infty)$ defined by

$$\begin{gathered} \Phi_{q,p}(x) = \frac{x_1 \partial_1 q(p_1 x_1, p_2 x_2)}{x_2 \partial_2 q(p_1 x_1, p_2 x_2)}, \\ X_{q,p} = \big\{(x_1, x_2) \in X \colon q(p_1 x_1, p_2 x_2) = 1\big\}, \end{gathered} \tag{2.10}$$

is bijective.

These conditions are satisfied, in particular, by functions q_α, $\alpha \in \mathbb{R} \setminus 0$.

In the above interpretation of transform (1.2) in the setting of the generalized model of pure industry functions q_α, $\alpha \leq 1$, correspond to unit cost functions for production systems with constant coefficient of elasticity of substitution of production factors whereas concave functions q different from q_α, $\alpha \leq 1$, and satisfying assumptions Q1, Q2 correspond to production systems with variable coefficient of elasticity of substitution.

Let us sketch the proof of statements 1, 2 ("if" part) and 3 of Theorem 2.1 in the case of smooth functions q satisfying Q1, Q2, Q3, see also the proof of Theorem 2.1.

Denote by R_q the generalized Radon transform defined by formula (1.2) using function q instead of q_α. Denote by C_q the corresponding canonical relation.

The proof of statement 1 of Theorem 2.1 for this class of functions q remains the same as for functions q_α, see Section 3.

It follows from Q2 that for any set $M \subseteq T^*X$, if $(p, \eta dp) \in C_q \circ M$, $\eta = (\eta_1, \eta_2)$, then $\eta_1 \eta_2 > 0$, where

$$C_q \circ M \overset{\text{def}}{=\!=} \big\{(p, \eta dp) \in T^*P \colon \exists (x, \xi dx) \in M \colon (p, \eta dp; x, \xi dx) \in C_q\big\}. \tag{2.11}$$

This implies statement 3 of Theorem 2.1 if we take into account the inclusion $WF(R_q u) \subseteq C_q \circ WF(u)$, $u \in \mathscr{E}'(X)$.

It follows from assumptions Q1, Q2, Q3 that the canonical projection from C_q to T^*X is injective and that the following inclusion holds:

$$C_q \circ WF(u) \cap T_p^* P \subseteq C_q \circ \dot{N}^* X_{q,p}. \tag{2.12}$$

Using (2.12) we can obtain the "if" part of statement 2 of Theorem 2.1.

The "only if" part of statement 2 of Theorem 2.1 is the most difficult to prove. To generalize it to the case of smooth functions q satisfying Q1, Q2, Q3 we need to obtain an analogue of Lemma 3.1 below. Restrictions in space and time prevent us from obtaining this analogue in the present paper. Nevertheless, we prove the main part of Lemma 3.1 (see also Lemma 4.1) for the general case of smooth functions q satisfying Q1, Q2 to be able to use it in a subsequent paper.

Remark 2.3. It follows from definitions (1.2), (2.1) that:

1. even if u is compactly supported in X, $v = R_\alpha u$ doesn't have compact support in general;
2. if v is not compactly supported in P, the value of $R_\alpha^* v$ is not always defined.

Hence $R_\alpha^* R_\alpha u$ is not always defined for $u \in C_c^\infty(X)$.

In the Gelfand's approach to generalized Radon transforms via double fibrations (see, e.g., Ref. [GGS]) this corresponds to the fact that the canonical projection from Z_α to X is not a proper map. Furthermore, this problem can't be resolved even by considering transform R_α on functions which have supports in some fixed compact subset of X.

Therefore, we are not able to deduce the "only if" part of statement 2 of Theorem 2.1 from the microlocal regularity properties of operator $R_\alpha^* R_\alpha$ as it can be done when the canonical projection from the incidence relation to the source space is a proper map and the double fibration corresponding to the generalized Radon transform satisfies the so-called Bolker assumption, see, e.g., Refs. [Gu], [GS], [Qu1] for more details.

3 Proof of Theorem 2.1 and Proposition 2.1

3.1 Approximation of R_α by proper Fourier integral operators

In this subsection we show that R_α is the Fourier integral operator associated with C_α and we formulate an auxilary lemma, which is crucial in the proof of the "only if" part of statement 2 of Theorem 2.1.

Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ be fixed. For any $u \in C_c^\infty(X)$ the following chain of equalities holds:

$$\begin{aligned} R_\alpha u(p) &= \int_{X_{\alpha,p}} u(x)\omega_{\alpha,p} = \int_{\mathbb{R}} \delta(s-1) \int_{q_\alpha(p_1x_1, p_2x_2)=s} u(x)\omega_{\alpha,p}\, ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\theta(s-1)} \int_{q_\alpha(p_1x_1,p_2x_2)=s} u(x)\omega_{\alpha,p}\, d\theta ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_X e^{i\phi_\alpha(p,x,\theta)} u(x)\, dx d\theta, \end{aligned} \tag{3.1}$$

where $p \in P$ and $\phi_\alpha(p, x, \theta)$ is the real phase function:

$$\phi_\alpha(p, x, \theta) = \theta(q_\alpha(p_1x_1, p_2x_2) - 1), \quad p \in P, x \in X, \theta \in \mathbb{R} \setminus 0. \tag{3.2}$$

It follows from (3.1), (3.2) that R_α is a Fourier integral operator associated with C_α. Hence R_α is a linear continuous operator from $C_c^\infty(X)$ to $C^\infty(P)$ and from $\mathscr{E}'(X)$ to $\mathscr{D}'(P)$. Statement 1 of Theorem 2.1 is proved.

Formula (3.1) implies that R_α is not a proper operator. We are going to approximate R_α by proper Fourier integral operators.

Let $0<\delta<\frac{1}{2}$ be fixed. Chose $\chi_\delta \in C^\infty(0,+\infty)$ as in (2.6) and $\widetilde{\chi}_\delta \in C^\infty(\mathbb{R})$ such that

$$\widetilde{\chi}_\delta(t)=\begin{cases}1, & |t|<\frac{1}{2}\delta,\\ 0, & |t|>\delta.\end{cases}$$

Put

$$æ_{\alpha,\delta}(p,x)=\chi_\delta(p_1)\chi_\delta(p_2)\chi_\delta(x_1)\chi_\delta(x_2)\widetilde{\chi}_\delta(q_\alpha(p_1x_1,p_2x_2)-1), \tag{3.3}$$

where $p=(p_1,p_2)\in P$, $x=(x_1,x_2)\in X$.

We define the Fourier integral operator $R_{\alpha,\delta}$ by the following formula:

$$R_{\alpha,\delta}u(p)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_X e^{i\phi_\alpha(p,x,\theta)}æ_{\alpha,\delta}(p,x)u(x)\,dxd\theta, \quad p\in P, \tag{3.4}$$

where $u\in C_c^\infty(X)$ and ϕ_α is defined in (3.2).

In a similar way with formula (3.1) we can obtain that

$$R_{\alpha,\delta}u(p)=\int_{X_{\alpha,p}} æ_{\alpha,\delta}(p,x)u(x)\omega_{\alpha,p}, \quad p\in P. \tag{3.5}$$

In particular, the definition of $R_{\alpha,\delta}$ doesn't depend on the choice of function $\widetilde{\chi}_\delta$.

It follows from formula (3.4) that the Schwartz kernel $K_{\alpha,\delta}$ of operator $R_{\alpha,\delta}$ is given by formula

$$K_{\alpha,\delta}(p,x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\phi_\alpha(p,x,\theta)}æ_{\alpha,\delta}(p,x)\,d\theta. \tag{3.6}$$

Formulas (3.3), (3.6) imply that

$$\begin{gathered}\operatorname{supp} K_{\alpha,\delta}\subseteq\big\{(p_1,p_2;x_1,x_2)\in P\times X\colon\\ q_\alpha(p_1x_1,p_2x_2)\le\delta,\quad \delta\le x_j\le\delta^{-1},\ \delta\le p_j\le\delta^{-1},\ j=1,2\big\}.\end{gathered} \tag{3.7}$$

Hence, $K_{\alpha,\delta}$ is compact and $R_{\alpha,\delta}$ is a proper operator.

As a proper Fourier integral operator, $R_{\alpha,\delta}$ continuously maps $C_c^\infty(X)$ to $C_c^\infty(P)$ and $\mathscr{E}'(X)$ to $\mathscr{E}'(P)$. Note that the dual operator $R^*_{\alpha,\delta}$ is given by the same formula (3.4) (after changing x to p and vice versa). Hence, the composition $R^*_{\alpha,\delta}R_{\alpha,\delta}$ is a well-defined proper linear continuous operator on $C_c^\infty(X)$ and on $\mathscr{E}'(X)$. We will show that, in fact, $R^*_{\alpha,\delta}R_{\alpha,\delta}$ is a pseudo-differential operator and we will compute its principal symbol.

In order to formulate the following result we need to introduce the set Σ_ϵ, $0<\epsilon<1$:

$$\Sigma_\epsilon=\big\{(\xi_1,\xi_2)\in\mathbb{R}^2\colon \epsilon<\frac{\xi_1}{\xi_2}<\epsilon^{-1}\big\}. \tag{3.8}$$

Lemma 3.1. *Let $\alpha \in \mathbb{R} \setminus \{0,1\}$ and $0 < \delta < \frac{1}{2}$ be fixed. Then $R^*_{\alpha,\delta} R_{\alpha,\delta}$ is a classical proper pseudo-differential operator of order 1 on X with principal symbol $\sigma^P(x, \xi dx)$, $(x, \xi dx) \in T^*X$, given by formula*

$$\sigma^P_{\alpha,\delta}(x, \xi dx) = \begin{cases} \frac{2\pi}{|\alpha|} \exp\bigl(i(\pi/4) \cdot \operatorname{sgn}(1-\alpha) \cdot (1 - \operatorname{sgn} \xi_1)\bigr)(x_1 x_2)^{-\frac{1}{\beta}} \times \\ \quad \times |\xi_1 \xi_2|^{-\frac{1}{\beta}} |\xi x|^{1-\frac{2}{\alpha}} æ^2_{\alpha,\delta}(p^*(x,\xi), x), \quad \text{if } \xi_1 \xi_2 > 0, \\ 0, \quad \text{otherwise,} \end{cases} \tag{3.9}$$

where $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, $p^*(x, \xi) = (p^*_1(x, \xi), p^*_2(x, \xi))$, $\xi x = \xi_1 x_1 + \xi_2 x_2$, $1/\alpha + 1/\beta = 1$,

$$p^*_j(x, \xi) = |\xi x|^{-\frac{1}{\alpha}} |\xi_j|^{\frac{1}{\alpha}} x_j^{-\frac{1}{\beta}}, \quad j = 1, 2. \tag{3.10}$$

Besides, there exists $\epsilon = \epsilon(\alpha, \delta) > 0$ such that the full symbol $\sigma_{\alpha,\delta}(x, \xi dx)$ of $R^*_{\alpha,\delta} R_{\alpha,\delta}$ is zero for $x \in X$, $\xi \in \mathbb{R}^2 \setminus (0 \cup \Sigma_\epsilon \cup -\Sigma_\epsilon)$.

For the definition of classical pseudo-differential operator see, e.g., Ref. [Shu], Definition 3.5.

The biggest part of the present paper is devoted to the proof of Lemma 3.1. In Section 4 we will present and prove a version of method of stationary phase and in Section 5 we will apply this method to prove Lemma 3.1.

3.2 Final part of the proof of Theorem 2.1

We begin this subsection by proving a lemma which follows from Lemma 3.1 and which will be used in the proof of the "only if" part of statement 2 of Theorem 2.1. In the end of this subsection we prove statements 2, 3 of Theorem 2.1.

We need to introduce some notations. For canonical relation C_α defined in (2.4) the transposed canonical relation C^t_α is defined by the following formula:

$$\begin{aligned} C^t_\alpha &\stackrel{\text{def}}{=\!=} \{(x, \xi dx, p, \eta dp) \in T^*X \times T^*P \colon (p, \eta dp, x, \xi dx) \in C_\alpha\} = \\ &= \Bigl\{(x_1, x_2; \lambda \sum\nolimits_{j=1}^2 p_j^\alpha x_j^{\alpha-1} dx_j; p_1, p_2; -\lambda \sum\nolimits_{j=1}^2 x_j^\alpha p_j^{\alpha-1} dp_j) \in \\ &\qquad \in T^*X \times T^*P \colon q_\alpha(p_1 x_1, p_2 x_2) = 1,\ \lambda \in \mathbb{R} \setminus 0\Bigr\}. \end{aligned} \tag{3.11}$$

In a similar way with (2.11), for a subset $M' \subseteq T^*P$ we denote

$$C^t_\alpha \circ M' \stackrel{\text{def}}{=\!=} \bigl\{(x, \xi dx) \in T^*X \colon \exists (p, \eta dp) \in M' \colon (x, \xi dx; p, \eta dp) \in C^t_\alpha\bigr\}. \tag{3.12}$$

We also need to introduce the following sets:

$$\begin{aligned} \Sigma &= \bigl\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \colon \xi_1 > 0,\ \xi_2 > 0\bigr\}, \\ T^*_\pm X &= \bigl\{(x, \xi dx) \in T^*X \colon \xi \in \pm\Sigma\bigr\}, \\ T^*_\pm P &= \bigl\{(p, \eta dp) \in T^*P \colon \eta \in \pm\Sigma\bigr\}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} X_\delta &= \bigl\{(x_1, x_2) \in X \colon \delta < x_j < \delta^{-1},\ j = 1, 2\bigr\}, \quad 0 < \delta < 1, \\ P_\delta &= \bigl\{(p_1, p_2) \in P \colon \delta < p_j < \delta^{-1},\ j = 1, 2\bigr\}, \quad 0 < \delta < 1. \end{aligned} \tag{3.14}$$

Lemma 3.2. *Let $\alpha \in \mathbb{R} \setminus \{0,1\}$ and $u \in \mathscr{E}'(X)$ be fixed. Then*

$$WF(u) \cap \big(T^*_+X \cup T^*_-X\big) \subseteq C^t_\alpha \circ WF(R_\alpha u). \tag{3.15}$$

Proof. Let $u \in \mathscr{E}'(X)$ and $(x^0, \xi^0 dx) \in WF(u)$ with $\xi^0 \in \Sigma \cup (-\Sigma)$ be fixed. Lemma 3.2 will be proved if we show that

$$(x^0, \xi^0 dx) \in C^t_\alpha \circ WF(R_\alpha u). \tag{3.16}$$

Chose $0 < \delta < \frac{1}{2}$ such that

$$\operatorname{supp} u \subseteq X_{2\delta} \text{ and } p^*(x^0, \xi^0) \in P_{2\delta}, \tag{3.17}$$

where $p^*(x^0,\xi^0) = (p_1^*(x^0,\xi^0), p_2^*(x^0,\xi^0))$ is defined in (3.10).

If follows from (3.3), (3.9), (3.17) that $\text{æ}_{\alpha,\delta}(p^*(x^0,\xi^0), x^0) = 1$ and

$$\sigma^P_{\alpha,\delta}(x^0, \xi^0 dx) \neq 0. \tag{3.18}$$

Formula (3.18) implies that

$$(x^0, \xi^0 dx) \in WF\big(R^*_{\alpha,\delta} R_{\alpha,\delta} u\big). \tag{3.19}$$

Operator $R^*_{\alpha,\delta}$ is a Fourier integral operator associated with canonical relation C^t_α and the following inclusion holds:

$$WF\big(R^*_{\alpha,\delta} R_{\alpha,\delta} u\big) \subseteq C^t_\alpha \circ WF(R_{\alpha,\delta} u). \tag{3.20}$$

It follows from formulas (2.6), (3.3), (3.5), (3.17) that

$$(R_{\alpha,\delta} u)(p) = \chi_\delta(p_1)\chi_\delta(p_2)(R_\alpha u)(p), \quad p \in P.$$

Hence

$$WF(R_{\alpha,\delta} u) \subseteq WF(R_\alpha u). \tag{3.21}$$

Formula (3.16) follows from (3.19), (3.20), (3.21). Lemma 3.2 is proved. $\square$

Proof of statements 2, 3 of Theorem 2.1. We denote by π_P, π_X, $\widetilde{\pi}_P$, $\widetilde{\pi}_X$ the canonical projections:

$$\begin{tikzcd} & C_\alpha \arrow[dl, "\pi_P"'] \arrow[dr, "\pi_X"] & \\ \dot{T}^*P & & \dot{T}^*X \end{tikzcd} \qquad \begin{tikzcd} & C^t_\alpha \arrow[dl, "\widetilde{\pi}_X"'] \arrow[dr, "\widetilde{\pi}_P"] & \\ \dot{T}^*X & & \dot{T}^*P \end{tikzcd}$$

As above, the dot means that we exclude the zero section.

Note that the maps π_P, π_X, $\widetilde{\pi}_P$, $\widetilde{\pi}_X$ are injective. Consider, for example, the map π_P. By definition

$$\begin{aligned} \pi_P\big(p_1, p_2; \lambda \textstyle\sum_{j=1}^2 x_j^\alpha p_j^{\alpha-1} dp_j; x_1, x_2; -\lambda \textstyle\sum_{j=1}^2 p_j^\alpha x_j^{\alpha-1} dx_j\big) \\ = \big(p_1, p_2; \lambda \textstyle\sum_{j=1}^2 x_j^\alpha p_j^{\alpha-1} dp_j\big). \end{aligned} \tag{3.22}$$

It follows from (3.22) that the value of $\pi_P(p, \eta dp, x, \xi dx)$ determines uniquely the ratio x_1/x_2. Since $x_1^\alpha p_1^\alpha + x_2^\alpha p_2^\alpha = 1$, it also determines x_1, x_2. Hence, it also determines λ and the map π_P is injective.

The following formulas follow from (2.2), (2.3), (2.4), (3.11):

$$\pi_P \circ \pi_X^{-1}(WF(u)) \subseteq T_+^*P \cup T_-^*P, \tag{3.23}$$

$$\pi_P \circ \pi_X^{-1}(\dot{N}^* X_{\alpha,p}) = T_{+,p}^*P \cup T_{-,p}^*P, \tag{3.24}$$

$$\widetilde{\pi}_X \circ \widetilde{\pi}_P^{-1}(T_p^*X \setminus 0) = \dot{N}^* X_{\alpha,p}. \tag{3.25}$$

where $p \in P$, $T_\pm^*P$ is defined in (3.13) and $T_{\pm,p}^* = T_\pm^*P \cap T_p^*P$.

Besides, using definitions (2.4), (3.11), (2.11), (3.12) we obtain that

$$\pi_P \circ \pi_X^{-1}(WF(u)) = C_\alpha \circ WF(u), \tag{3.26}$$

$$\widetilde{\pi}_X \circ \widetilde{\pi}_P^{-1}(WF(R_\alpha u)) = C_\alpha^t \circ WF(R_\alpha u). \tag{3.27}$$

Statement 3. Using (3.23), (3.26) and inclusion $WF(R_\alpha u) \subseteq C_\alpha \circ WF(u)$ we obtain that if $(x, \xi dx) \in WF(R_\alpha u)$, $\xi = (\xi_1, \xi_2)$, then $\xi_1 \xi_2 > 0$. Statement 3 of Theorem 2.1 is proved.

Statement 2: "if" part. Suppose that (2.5) holds. Using injectivity of π_P we obtain that

$$\pi_P \circ \pi_X^{-1}(WF(u)) \cap \pi_P \circ \pi_X^{-1}(\dot{N}^* X_{\alpha,p}) = \varnothing.$$

This formula together with formulas (3.23), (3.24), (3.26) imply that

$$C_\alpha \circ WF(u) \cap T_p^*P = \varnothing. \tag{3.28}$$

Using formula (3.28) and inclusion $WF(R_\alpha u) \subseteq C_\alpha \circ WF(u)$ we obtain that $p \notin \operatorname{sing\,supp} R_\alpha u$.

Statement 2: "only if" part. Suppose that $p \notin \operatorname{sing\,supp} R_\alpha u$. Equivalently,

$$WF(R_\alpha u) \cap (T_p^*P \setminus 0) = \varnothing. \tag{3.29}$$

Using injectivity of $\widetilde{\pi}_X$ and formulas (3.25), (3.27) we obtain from (3.29) the following formula:

$$C_\alpha^t \circ WF(R_\alpha u) \cap \dot{N}^* X_{\alpha,p} = \varnothing. \tag{3.30}$$

Note that formula (2.3) implies that

$$\dot{N}^* X_{\alpha,p} \subseteq T_+^*X \cup T_-^*X. \tag{3.31}$$

Using inclusion (3.15) of Lemma 3.2 and formulas (3.30), (3.31), we obtain formula (2.5). Theorem 2.1 is proved. $\square$

3.3 Proof of Proposition 2.1

Let $u \in \mathscr{E}'(X)$ and $(x, \xi dx) \in T^*X$ satisfying the conditions of Proposition 2.1 be fixed. It follows from the definition of δ and from (3.10) that

$$\sigma_{\alpha,\delta}^P(x, \xi dx) \neq 0, \tag{3.32}$$

where $\sigma^P_{\alpha,\delta}$ is the symbol of operator $R^*_{\alpha,\delta}R_{\alpha,\delta}$ defined in (3.9).

"Only if" part. Suppose that $(x,\xi dx)\in WF(u)$. Then it follows from (3.32) that

$$(x,\xi dx)\in WF\big(R^*_{\alpha,\delta}R_{\alpha,\delta}u\big). \tag{3.33}$$

Since u satisfies (2.8) and $2\delta<\varepsilon$, the following formula is valid:

$$R_{\alpha,\delta}u=\chi^1_\delta\chi^2_\delta R_\alpha u. \tag{3.34}$$

Also note that

$$R^*_{\alpha,\delta}=\tilde\chi^1_\delta\tilde\chi^2_\delta R^*_\alpha\chi^1_\delta\chi^2_\delta, \tag{3.35}$$

where $\tilde\chi^j_\delta$ is the operator of multiplication by $\chi_\delta(x_j)$, $j=1$, 2.

Formulas (3.34), (3.35) imply that

$$WF\big(R^*_{\alpha,\delta}R_{\alpha,\delta}u\big)\subseteq WF\big(R^*_\alpha(\chi^1_\delta)^2(\chi^2_\delta)^2R_\alpha u\big).$$

From this formula, taking into account that we can replace $(\chi^j_\delta)^2$ by χ^j_δ, $j=1$, 2, and from formula (3.33) it follows that

$$(x,\xi dx)\in WF\big(R^*_\alpha\chi^1_\delta\chi^2_\delta R_\alpha u\big). \tag{3.36}$$

"If" part. Suppose now that (3.36) holds. Then we have the following sequence of inclusions:

$$\begin{gathered}(x,\xi dx)\in WF\big(R^*_\alpha\chi^1_\delta\chi^2_\delta R_\alpha u\big)\\ \subseteq C^t_\alpha\circ WF(\chi^1_\delta\chi^2_\delta R_\alpha u)\subseteq C^t_\alpha\circ WF(R_\alpha u)\\ \subseteq C^t_\alpha\circ C_\alpha\circ WF(u)\subseteq WF(u),\end{gathered}$$

since R^*_α is the Fourier integral operator associated with canonical relation C^t_α and the inclusion $C^t_\alpha\circ C_\alpha\subseteq \operatorname{diag}(T^*X\times T^*X)$ holds, where

$$\operatorname{diag}(T^*X\times T^*X)\overset{\text{def}}{=\!=}\big\{(x,\xi dx,x,\xi dx)\in T^*X\times T^*X\big\}, \tag{3.37}$$

$$\begin{gathered}C^t_\alpha\circ C_\alpha\overset{\text{def}}{=\!=}\big\{(y,\zeta dy,x,\xi dx)\in T^*X\times T^*X\colon\\ \exists(p,\eta dp)\in T^*P\colon(y,\zeta dy,p,\eta dp)\in C^t_\alpha,\ (p,\eta dp,x,\xi dx)\in C_\alpha\big\}.\end{gathered} \tag{3.38}$$

Proposition 2.1 is proved.

4 Method of stationary phase

In this section we formulate and prove a version of method of stationary phase which will be used in Section 5 in the proof of Lemma 3.1.

We say that a subset $\widehat\Sigma\subseteq\mathbb{R}^2\setminus 0$ is conic if

$$\big\{t\xi\in\mathbb{R}^2\colon t>0,\ \xi\in\widehat\Sigma\big\}=\widehat\Sigma.$$

Let $\widehat{\Sigma} \subseteq \mathbb{R}^2 \setminus 0$ be an open conic set. We denote by $S^\gamma(X \times \widehat{\Sigma})$, $\gamma \in \mathbb{R}$, the set of functions $a \in C^\infty(X \times \widehat{\Sigma})$ satisfying the following property: for any compact $K \subset X \times \widehat{\Sigma}$ and for any $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$ there exists such constant $C_K^{\alpha,\beta} > 0$ that

$$\left|D_x^\alpha D_\xi^\beta a(x, \xi)\right| \leq C_K^{\alpha,\beta} |\xi|^{\gamma - |\beta|}, \quad |\xi| \geq 1,\ \xi = t\widetilde{\xi},\ t > 0,\ (x, \widetilde{\xi}) \in K,$$

where

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad D_\xi^\beta = \frac{\partial^{|\beta|}}{\partial \xi_1^{\beta_1} \partial \xi_2^{\beta_2}},$$

and $|\alpha| = \alpha_1 + \alpha_2$, $|\beta| = \beta_1 + \beta_2$.

The space $S^\gamma(P \times \widehat{\Sigma})$, $\gamma \in \mathbb{R}$, is defined in a similar way. We also put $S^{-\infty}(X \times \widehat{\Sigma}) = \cap_{\gamma \in \mathbb{R}} S^\gamma(X \times \widehat{\Sigma})$ and $S^{-\infty}(P \times \widehat{\Sigma}) = \cap_{\gamma \in \mathbb{R}} S^\gamma(P \times \widehat{\Sigma})$.

We will formulate and prove a version of method of stationary phase for the integrals of the following form:

$$I(p, \xi) = \int_{X_{q,p}} e^{iS(x,\xi)} f(x, \xi)\, \omega_{q,p}, \tag{4.1}$$

where $q \in C^\infty(X)$ satisfies Q1, Q2 of Remark 2.2, $X_{q,p}$ is defined in (2.10) with orientation given by the volume form $|\nabla_x q(p_1 x_1, p_2 x_2)| \omega_{q,p}$, where $\omega_{q,p}$ is the Gelfand–Leray form:

$$\omega_{q,p} = d_x q(p_1 x_1, p_2 x_2) \lrcorner (dx_1 \wedge dx_2),$$

and $S \in C^\infty(X \times \widehat{\Sigma})$ satisfies the following assumption:

S1. $S(x, \lambda\xi) = \lambda S(x, \xi)$, $x \in X$, $\xi \in \widehat{\Sigma}$, $\lambda > 0$.

Assumtion Q2 of Remark 2.2 implies that at fixed $p \in P$ the curve $X_{q,p}$ is the graph of C^∞ function

$$x_2 = x_2(x_1, p), \quad x_1^-(p) < x_1 < x_1^+(p) \leq +\infty. \tag{4.2}$$

Put $\widetilde{S}(t, p, \xi) = S((t, x_2(t, p)), \xi)$.

We suppose that q and S satisfy the following additional assumptions:

QS1. at fixed $p \in P$, $\xi \in \widehat{\Sigma}$ function $\widetilde{S}(\cdot, p, \xi)$ has the unique local extremum $x_1^*(p, \xi)$ on the interval $(x_1^-(p), x_1^+(p))$ which is the unique zero of function $\widetilde{S}_t(\cdot, p, \xi)$; the map $x_1^* \colon X \times \widehat{\Sigma} \to (0, +\infty)$ is C^∞ and open;

QS2. the second derivative $\widetilde{S}_{tt}(x_1^*(p, \xi), p, \xi)$ is nonzero and has the same sign for all $p \in P$, $\xi \in \widehat{\Sigma}$.

Lemma 4.1. *Let $\widehat{\Sigma}$ be Σ or $-\Sigma$. Let $q \in C^\infty(X)$ satisfy assumptions Q1, Q2 of Remark 2.2 and $S \in C^\infty(X \times \widehat{\Sigma})$ satisfy assumption S1 above. Suppose also that q, S satisfy assumptions QS1, QS2 defined above. Suppose that $f \in S^m(X \times \widehat{\Sigma})$, $m \in \mathbb{R}$, and $f(\cdot, \xi)$ has compact support in X for any fixed $\xi \in \widehat{\Sigma}$. Then:*

1. *the following inclusion holds:*

$$e^{-iS^*(p,\xi)} I(p,\xi) \in S^{m-\frac{1}{2}}(P \times \widehat{\Sigma}), \tag{4.3}$$

where $I(p,\xi)$ *is defined in* (4.1), $S^*(p,\xi) = S(x^*(p,\xi),\xi)$, $x^*(p,\xi) = (x_1^*(p,\xi), x_2^*(p,\xi))$, x_1^* *is defined in assumption QS1 above,* $x_2^*(p,\xi) = x_2(x_1^*(p,\xi),p)$*;*

2. *if, in addition, function* f *satisfies*

$$f(x,\lambda\xi) = \lambda^m f(x,\xi), \quad x \in X,\ \xi \in \widehat{\Sigma},\ \lambda > 0, \tag{4.4}$$

then there exist the unique functions $a_j \in C^\infty(P \times \widehat{\Sigma})$, $j \in \mathbb{N} \cup 0$, *satisfying*

$$a_j(p,\lambda\xi) = \lambda^{m-\frac{1}{2}-j} a_j(p,\xi), \quad p \in P,\ \xi \in \widehat{\Sigma},\ \lambda > 0,\ j \in \mathbb{N} \cup 0, \tag{4.5}$$

and such that for any $N \in \mathbb{N} \cup 0$ *the following formula is valid:*

$$e^{-iS^*(p,\xi)} I(p,\xi) - \sum\nolimits_{j=0}^{N} a_j(p,\xi) = r_N(p,\xi) \in S^{m-N-\frac{3}{2}}(P \times \widehat{\Sigma}); \tag{4.6}$$

furthermore, the following formula is holds:

$$\begin{aligned} a_0(p,\xi) = (2\pi)^{1/2} e^{i\pi/4} f(x^*(p,\xi),\xi) \big(\widetilde{S}_{tt}(x_1^*(p,\xi),p,\xi)\big)^{-1/2} \times \\ \times p_2^{-1} \big(\partial_2 q\big(p_1 x_1^*(p,\xi), p_2 x_2^*(p,\xi)\big)\big)^{-1}. \end{aligned} \tag{4.7}$$

Proof. We suppose that $\widehat{\Sigma} = \Sigma$ and

$$\widetilde{S}_{tt}(x_1^*(p,\xi),p,\xi) > 0, \quad p \in P,\ \xi \in \Sigma. \tag{4.8}$$

The other three cases ($\widehat{\Sigma} = -\Sigma$, $\widetilde{S}_{tt} > 0$; $\widehat{\Sigma} = \Sigma$, $\widetilde{S}_{tt} < 0$; $\widehat{\Sigma} = -\Sigma$, $\widetilde{S}_{tt} < 0$) can be considered in a similar way.

The proof of Lemma 4.1 consists of six steps.

Step 1: uniform Morse coordinate. Using integration by parts and taking into account that $\widetilde{S}_t(x_1^*(p,\xi),p,\xi) = 0$ we obtain that at fixed $p \in P$, $\xi \in \Sigma$ for x_1 sufficiently close to $x_1^*(p,\xi)$ the following chain of equalities holds:

$$\begin{aligned} &\widetilde{S}(x_1,p,\xi) - S^*(p,\xi) \\ &= \int_0^1 (1-\tau) \frac{d^2}{d\tau^2} \widetilde{S}\big(x_1^*(p,\xi) + \tau(x_1 - x_1^*(p,\xi)),p,\xi\big)\, d\tau \\ &= (x_1 - x_1^*(p,\xi))^2 \int_0^1 (1-\tau) \widetilde{S}_{tt}\big(x_1^*(p,\xi) + \tau(x_1 - x_1^*(p,\xi),p,\xi\big) d\tau \\ &= \frac{1}{2} z_1^2(x_1,p,\xi), \end{aligned} \tag{4.9}$$

where $z_1(x_1,p,\xi)$ is defined for x_1 sufficiently close to $x_1^*(p,\xi)$:

$$\begin{aligned} z_1(x_1,p,\xi) = \sqrt{2}(x_1 - x_1^*(p,\xi)) \times \\ \times \Big(\int_0^1 (1-\tau) \widetilde{S}_{tt}\big(x_1^*(p,\xi) + \tau(x_1 - x_1^*(p,\xi)),p,\xi\big) d\tau \Big)^{1/2}. \end{aligned} \tag{4.10}$$

Let $0 < \epsilon < 1$ be fixed. Define $s = x_1 - x_1^*(p, \xi)$ for $x_1 \in (x_1^-(p), x_1^+(p))$.

Recall that the sets Σ_ϵ, P_ϵ, $\epsilon > 0$, are defined in formulas (3.8), (3.14). Since x_1^* is open and continuous, the set $x_1^*(P_\epsilon \times \Sigma_\epsilon)$ in an interval and $x_1^*(P_\epsilon \times \Sigma_\epsilon) \Subset (x_1^-(p), x_1^+(p))$.

Hence, for all $p \in P_\epsilon$ and $\xi \in \Sigma_\epsilon$ the range of variable s contains an interval $(-\widetilde{\nu}_\epsilon, \widetilde{\nu}_\epsilon)$, $\widetilde{\nu}_\epsilon > 0$, not depending on p, ξ.

Fix $\widetilde{p} \in P$ and $\widetilde{\xi} \in \Sigma$. It follows from (4.8), (4.10) that

$$\frac{\partial z_1}{\partial x_1}\big(x_1^*(\widetilde{p}, \widetilde{\xi}), \widetilde{p}, \widetilde{\xi}\big) = \big(\widetilde{S}_{tt}(x_1^*(\widetilde{p}, \widetilde{\xi}), \widetilde{p}, \widetilde{\xi})\big)^{1/2} > 0. \tag{4.11}$$

Since the derivatives $S_{tt}(x_1, p, \xi)$ and $\partial z_1/\partial x_1(x_1, p, \xi)$ are continuous w.r.t. (x_1, p, ξ), there exist open neighborhood $U_1(\widetilde{p}, \widetilde{\xi})$ of point $0 \in \mathbb{R}$, open neighborhood $U_2(\widetilde{p}, \widetilde{\xi})$ of point $\widetilde{p} \in P$ and open conic neighborhood $U_3(\widetilde{p}, \widetilde{\xi})$ of point $\widetilde{\xi} \in \Sigma$ such that for any $(s, p, \xi) \in \Pi_{j=1}^3 U_j(\widetilde{p}, \widetilde{\xi})$ function $z_1(x_1, p, \xi)$ is well-defined and the following inequality holds:

$$\frac{\partial z_1}{\partial x_1}\big(s + x_1^*(p, \xi), p, \xi\big) > 0.$$

The set $\overline{P}_\epsilon$ is compact and the set $\overline{\Sigma}_\epsilon$ is conically compact (i.e., it is conic and its intersection with the unit circle S^1 in $\mathbb{R}^2$ is compact). The system of sets $\Pi_{j=2}^3 U_j(\widetilde{p}, \widetilde{\xi})$, $\widetilde{p} \in P$, $\widetilde{\xi} \in \Sigma$, is an open cover of $\overline{P}_\epsilon \times \overline{\Sigma}_\epsilon$. Let $\Pi_{j=2}^3 U_j(\widetilde{p}^k, \widetilde{\xi}^k)$, $k = 1, \ldots, N_1$, be its finite subcover. Denote by $(-\nu_\epsilon, \nu_\epsilon)$ any symmetric with respect to zero interval contained in $\cap_{k=1}^{N_1} U_1(\widetilde{p}^k, \widetilde{\xi}^k)$. Then $z_1(x_1, p, \xi)$ is well-defined for $x_1 - x_1^*(p, \xi) \in (-\nu_\epsilon, \nu_\epsilon)$, $p \in P_\epsilon$, $\xi \in \Sigma_\epsilon$ and

$$\frac{\partial z_1}{\partial x_1}(x_1, p, \xi) > 0 \text{ for } x_1 - x_1^*(p, \xi) \in (-\nu_\epsilon, \nu_\epsilon),\ p \in P_\epsilon,\ \xi \in \Sigma_\epsilon. \tag{4.12}$$

Step 2: partition of unity. Fix $\widetilde{p} \in P_\epsilon$, $\widetilde{\xi} \in \Sigma_\epsilon$. Since x_1^* is continuous, there exist an open neighborhood $\widetilde{P}(\widetilde{p})$ of $\widetilde{p}$ in P_ϵ and an open conic neighborhood $\widetilde{\Sigma}(\widetilde{\xi})$ of $\widetilde{\xi}$ in Σ_ϵ such that $|x_1^*(\widetilde{P}(\widetilde{p}) \times \widetilde{\Sigma}(\widetilde{\xi}))| < \frac{1}{2}\nu_\epsilon$.

The set $\overline{P}_{2\epsilon}$ is compact and the set $\overline{\Sigma}_{2\epsilon}$ is conically compact. The sets $\widetilde{P}(\widetilde{p}) \times \widetilde{\Sigma}(\widetilde{\xi})$, $\widetilde{p} \in P_\epsilon$, $\widetilde{\xi} \in \Sigma_\epsilon$, form an open cover of $\overline{P}_{2\epsilon} \times \overline{\Sigma}_{2\epsilon}$. Let $\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k)$, $k = 1, \ldots, N_2$ be its finite subcover, so that

$$P_{2\epsilon} \times \Sigma_{2\epsilon} \subseteq \bigcup_{k=1}^{N_2} \big(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k)\big) \subseteq P_\epsilon \times \Sigma_\epsilon. \tag{4.13}$$

Denote $(x_{1,k}^-, x_{1,k}^+) = x_1^*(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k))$, $k = 1, \ldots, N_2$. Then

$$x_{1,k}^+ - x_{1,k}^- < \tfrac{1}{2}\nu_\epsilon, \quad k = 1, \ldots, N_2. \tag{4.14}$$

It follows from (4.12), (4.14) that

$$\frac{\partial z_1}{\partial x_1}(x_1, p, \xi) > 0 \text{ on } (x_{1,k}^- - \tfrac{1}{2}\nu_\epsilon, x_{1,k}^+ + \tfrac{1}{2}\nu_\epsilon) \times \widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k). \tag{4.15}$$

Fix $k \in \{1, \ldots, N_2\}$. Let $\varphi_1^k \in C^\infty(0, +\infty)$ satisfy $0 \le \varphi_1^k(x_1) \le 1$, $x_1 > 0$,

$$\varphi_1^k(x_1) = \begin{cases} 1, & \text{if } x_{1,k}^- - \frac{1}{8}\nu_\epsilon < x_1 < x_{1,k}^+ + \frac{1}{8}\nu_\epsilon, \\ 0, & \text{if } x_1 < x_{1,k}^- - \frac{1}{4}\nu_\epsilon \text{ or } x_1 > x_{1,k}^+ + \frac{1}{4}\nu_\epsilon. \end{cases} \tag{4.16}$$

We define $\varphi_2^k = 1 - \varphi_1^k$. Formula (4.16) implies that

$$\operatorname{dist}\bigl(\operatorname{supp}\varphi_2^k, x_1^*(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k))\bigr) \ge \tfrac{1}{8}\nu_\epsilon. \tag{4.17}$$

At fixed $k \in \{1, \ldots, N_2\}$, $p \in P$, $\xi \in \Sigma$ we rewrite integral (4.1) in the following form:

$$I(p, \xi) = I_1^k(p, \xi) + I_2^k(p, \xi), \tag{4.18}$$

$$I_j^k(p, \xi) = \int_{x_1^-(p)}^{x_1^+(p)} \frac{e^{i\widetilde{S}(x_1, p, \xi)} f(x_1, x_2(x_1, p), \xi)\varphi_j^k(x_1)\, dx_1}{p_2 \partial_2 q(p_1 x_1, p_2 x_2(x_1, p))}, \quad j = 1, 2, \tag{4.19}$$

where $\partial_2 q(y_1, y_2) = \frac{\partial q}{\partial y_2}(y_1, y_2)$.

Step 3: integral $I_1^k(p, \xi)$. It follows from (4.15) that at fixed $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$ function $z_1(\cdot, p, \xi)$ on $(x_{1,k}^- - \frac{1}{2}\nu_\epsilon, x_{1,k}^+ + \frac{1}{2}\nu_\epsilon)$ has the inverse $x_1(\cdot, p, \xi)$.

It follows from (4.9), (4.16), (4.19) that

$$I_1^k(p, \xi) = e^{iS^*(p, \xi)} \int_{-\infty}^{+\infty} e^{\frac{i}{2} z_1^2} g^k(z_1, p, \xi)\, dz_1, \tag{4.20}$$

where

$$g^k(z_1, p, \xi) = \begin{cases} f(x_1(z_1, p, \xi), x_2(z_1, p, \xi), \xi) \frac{\partial x_1}{\partial z_1}(z_1, p, \xi) \times \\ \qquad \times p_2^{-1}\bigl(\partial_2 q(p_1 x_1(z_1, p, \xi), p_2 x_2(z_1, p, \xi))\bigr)^{-1} \times \\ \qquad\qquad \times \varphi_1^k(x_1(z_1, p, \xi)), \quad \text{if } x_1(z_1, p, \xi) \in \operatorname{supp}\varphi_1^k, \\ 0, \quad \text{otherwise}, \end{cases} \tag{4.21}$$

and $x_2(z_1, p, \xi) = x_2(x_1(z_1, p, \xi), p)$.

Define $\widetilde{z}_1(x_1, p, \xi) = |\xi|^{-\frac{1}{2}} z_1(x_1, p, \xi)$. It follows from assumption S1 and from formula (4.10) that $\widetilde{z}_1(x_1, p, \lambda\xi) = \widetilde{z}_1(x_1, p, \xi)$, $x_1 \in (x_{1,k}^- - \frac{1}{2}\nu_\epsilon, x_{1,k}^+ + \frac{1}{2}\nu_\epsilon)$, $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $\lambda > 0$. Note that at fixed $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$ the inverse $\widetilde{x}_1(\cdot, p, \xi)$ of $\widetilde{z}_1(\cdot, x, \xi)$ on $(x_{1,k}^- - \frac{1}{2}\nu_\epsilon, x_{1,k}^+ + \frac{1}{2}\nu_\epsilon)$ exists and the following formulas hold:

$$x_1(z_1, p, \xi) = \widetilde{x}_1(|\xi|^{-\frac{1}{2}} z_1, p, \xi), \tag{4.22}$$

$$\widetilde{x}_1(\widetilde{z}_1, p, \lambda\xi) = \widetilde{x}_1(\widetilde{z}_1, p, \xi), \quad \lambda > 0, \tag{4.23}$$

for all z_1 (resp. $\widetilde{z}_1$) in the domain of definition of $x_1(\cdot, p, \xi)$ (resp. $\widetilde{x}_1(\cdot, p, \xi)$), $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$.

It follows from assumption S1 and from formula (4.10) that the quantity $|\xi|^{-\frac{1}{2}} z_1(x_1, p, \xi)$ is uniformly bounded for $x_1 \in \operatorname{supp}\varphi_1^k$, $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$.

Taking this into account and using formulas (4.22), (4.23) we obtain that at fixed $k \in \{1,\dots,N_2\}$ for any α, $\beta \in \mathbb{Z}_+^2$ there exist constants $C_{1,k}^{\alpha,\beta} > 0$ such that the following inequalities hold:

$$
\begin{aligned}
&\big|D_p^\alpha D_\xi^\beta x_1(z_1,p,\xi)\big|,\ \big|D_p^\alpha D_\xi^\beta x_2(z_1,p,\xi)\big|, \big|D_p^\alpha D_\xi^\beta \varphi_1^k(x_1(z_1,p,\xi))\big|,\\
&\qquad \big|D_p^\alpha D_\xi^\beta \big(q(p_1x_1(z_1,p,\xi),p_2x_2(z_1,p,\xi))\big)^{-1}\big| \le C_{1,k}^{\alpha,\beta}|\xi|^{-|\beta|},\\
&\big|D_p^\alpha D_\xi^\beta \frac{\partial x_1}{\partial z_1}(z_1,x,\xi)\big| \le C_{1,k}^{\alpha,\beta}|\xi|^{-|\beta|-\frac{1}{2}},\\
&\big|D_p^\alpha D_\xi^\beta f(x_1(z_1,p,\xi),x_2(z_1,p,\xi),\xi)\big| \le C_{1,k}^{\alpha,\beta}|\xi|^{m-|\beta|},
\end{aligned}
\tag{4.24}
$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $|\xi| \ge 1$, $z_1 \in z_1(\operatorname{supp}\varphi_1^k,p,\xi)$.

Formulas (4.20), (4.21) and inequalities (4.24) imply that at fixed $k \in \{1,\dots,N_2\}$ for any α, $\beta \in \mathbb{Z}_+^2$ there exist constants $C_{2,k}^{\alpha,\beta} > 0$, $C_{3,k}^{\alpha,\beta} > 0$ such that

$$
\big|D_p^\alpha D_\xi^\beta g^k(z_1,p,\xi)\big| \le C_{2,k}^{\alpha,\beta}|\xi|^{m-\frac{1}{2}-|\beta|}, \tag{4.25}
$$

$$
\big|D_p^\alpha D_\xi^\beta \big(e^{-iS^*(p,\xi)} I_1^k(p,\xi)\big)\big| \le C_{3,k}^{\alpha,\beta}|\xi|^{m-\frac{1}{2}-|\beta|}. \tag{4.26}
$$

where $z_1 \in \mathbb{R}$, $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $|\xi| \ge 1$.

We need to introduce some notations. Let $\widetilde{P} \subseteq P$ be an open set and $\widetilde{\Sigma} \subseteq \Sigma$ be an open conic set. Let $\gamma \in \mathbb{R}$ be fixed. We denote by $\widetilde{S}^\gamma(\widetilde{P}\times\widetilde{\Sigma})$ the set of functions $a \in C^\infty(\widetilde{P}\times\widetilde{\Sigma})$ such that for any α, $\beta \in \mathbb{Z}_+^2$ there exists a constant $\widetilde{C}^{\alpha,\beta} > 0$ such that

$$
\big|D_p^\alpha D_\xi^\beta a(p,\xi)\big| \le \widetilde{C}^{\alpha,\beta}|\xi|^{\gamma-|\beta|}, \quad p \in \widetilde{P},\ \xi \in \widetilde{\Sigma},\ |\xi| \ge 1. \tag{4.27}
$$

Besides, we denote $\widetilde{S}^{-\infty}(\widetilde{P}\times\widetilde{\Sigma}) = \cap_{\gamma\in\mathbb{R}}\widetilde{S}^\gamma(\widetilde{P}\times\widetilde{\Sigma})$.

Note that we can rewrite formula (4.26) in the following form:

$$
e^{-iS^*(p,\xi)} I_1^k(p,\xi) \in \widetilde{S}^{m-\frac{1}{2}}(\widetilde{P}(\widetilde{p}^k)\times\widetilde{\Sigma}(\widetilde{\xi}^k)), \quad k = 1,\dots,N_2. \tag{4.28}
$$

Step 4: functions $a_j(p,\xi)$. Throughout the Steps 4, 5 we suppose that (4.4) holds.

Using assumptions Q1, S1 and formulas (4.4), (4.19), (4.20) we obtain the identity

$$
\int_{-\infty}^{+\infty} e^{\frac{i}{2}z_1^2} g^k(z_1,p,\lambda\xi)\,dz_1 = \lambda^m \int_{-\infty}^{+\infty} e^{\frac{i}{2}\lambda z_1^2} g^k(z_1,p,\xi)\,dz_1, \tag{4.29}
$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $\lambda > 0$.

Using the Plancherel identity we get

$$
\int_{-\infty}^{+\infty} e^{\frac{i}{2}\lambda z_1^2} g^k(z_1,p,\xi)\,dz_1 = (2\pi\lambda)^{-\frac{1}{2}} e^{i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{\frac{it^2}{2\lambda}} \widehat{g}^k(t,p,\xi)\,dt, \tag{4.30}
$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $\lambda > 0$ and

$$\widehat{g}^k(t,p,\xi) = \int_{-\infty}^{+\infty} e^{-itz_1} g^k(z_1,p,\xi)\,dz_1.$$

Note that the following formula holds for $t \in \mathbb{R}$, $\lambda > 0$, $N \in \mathbb{N} \cup 0$:

$$e^{\frac{it^2}{2\lambda}} = \sum_{j=0}^{N} \frac{1}{j!}\left(\frac{it^2}{2\lambda}\right)^j + \frac{1}{N!}\left(\frac{it^2}{2}\right)^{N+1} \int_0^{1/\lambda} e^{\frac{i}{2}st^2}\left(\frac{1}{\lambda} - s\right)^N ds.$$

Hence, the following equality is valid:

$$\int\limits_{-\infty}^{+\infty} e^{\frac{it^2}{2\lambda}} \widehat{g}^k(t,p,\xi)\,dt = \sum_{j=0}^{N} \frac{(2i\lambda)^{-j}}{j!} \int\limits_{-\infty}^{+\infty} (it)^{2j} \widehat{g}^k(t,p,\xi)\,dt + r_N^k(\lambda,p,\xi), \quad (4.31)$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $\lambda > 0$ and

$$r_N^k(\lambda,p,\xi) = \int\limits_{-\infty}^{+\infty} \int\limits_0^{1/\lambda} \frac{(it^2)^{N+1}}{N!2^{N+1}} e^{\frac{i}{2}st^2} \left(\frac{1}{\lambda} - s\right)^N \widehat{g}^k(t,p,\xi)\,ds\,dt. \quad (4.32)$$

Taking into account formulas (4.16), (4.21), we obtain the formula

$$\int_{-\infty}^{+\infty} (it)^{2j} \widehat{g}^k(t,p,\xi)\,dt = 2\pi \frac{\partial^{2j} g}{\partial z_1^{2j}}(0,p,\xi), \quad j = 0,\dots,N, \quad (4.33)$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$,

$$\begin{aligned} g(z_1,p,\xi) = f(x_1(z_1,p,\xi), x_2(z_1,p,\xi),\xi) \frac{\partial x_1}{\partial z_1}(z_1,p,\xi) \times \\ \times p_2^{-1}\big(\partial_2 q(p_1 x_1(z_1,p,\xi), p_2 x_2(z_1,p,\xi))\big)^{-1}. \end{aligned} \quad (4.34)$$

Note that functions $x_1(z_1,p,\xi)$ (resp. $\widetilde{x}_1(\widetilde{z}_1,p,\xi)$), $g(z_1,p,\xi)$ at fixed $p \in P$, $\xi \in \Sigma$ are defined, at least, for z_1 (resp. $\widetilde{z}_1$) in a small neighborhood of $0 \in \mathbb{R}$ and in this neighborhood formulas (4.22), (4.23) hold. This is a consequence of (4.11), (4.34).

Formulas (4.22), (4.23) imply that

$$\frac{\partial^s x_1}{\partial z_1^s}(0,p,\lambda\xi) = \lambda^{-\frac{s}{2}} \frac{\partial^s x_1}{\partial z_1^s}(0,p,\xi), \quad \lambda > 0, \quad (4.35)$$

where $p \in P$, $\xi \in \Sigma$, $s \in \mathbb{N} \cup 0$. Using (4.4), (4.34), (4.35) we obtain the identity

$$\frac{\partial^{2j} g}{\partial z_1^{2j}}(0,p,\lambda\xi) = \lambda^{m-j-\frac{1}{2}} \frac{\partial^{2j} g}{\partial z_1^{2j}}(0,p,\xi), \quad \lambda > 0,\ j \in \mathbb{N} \cup 0, \quad (4.36)$$

where $p \in P$, $\xi \in \Sigma$.

Define

$$a_j(p,\xi) = \frac{(2\pi)^{\frac{1}{2}}}{(2i)^j j!} e^{i\frac{\pi}{4}} \frac{\partial^{2j} g}{\partial z_1^{2j}}(0,p,\xi), \quad j = 0,\ldots,N, \tag{4.37}$$

where $p \in P$, $\xi \in \Sigma$.

It follows from (4.10), (4.11), (4.34), (4.36), (4.37) that $a_j \in C^\infty(P \times \Sigma)$, $j = 0, \ldots, N$, and that formulas (4.5), (4.7) hold.

Step 5: function $r_N^k(\lambda, p, \xi)$. Using formulas (4.20), (4.29), (4.30), (4.31), (4.37), we obtain the following equalities:

$$e^{-iS^*(p,\lambda\xi)} I_1^k(p,\lambda\xi) - \sum\nolimits_{j=0}^{N} a_j(p,\lambda\xi) = (2\pi)^{-\frac{1}{2}} e^{i\frac{\pi}{4}} r_N^k(1,p,\lambda\xi), \tag{4.38}$$

$$\begin{aligned} e^{-iS^*(p,\lambda\xi)} I_1^k(p,\lambda\xi) &- \sum\nolimits_{j=0}^{N} \lambda^{m-\frac{1}{2}-j} a_j(p,\xi) \\ &= (2\pi)^{-\frac{1}{2}} \lambda^{m-\frac{1}{2}} e^{i\frac{\pi}{4}} r_N^k(\lambda,p,\xi), \end{aligned} \tag{4.39}$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $\lambda > 0$, $N \in \mathbb{N} \cup 0$.

Formulas (4.5), (4.38), (4.39) imply that

$$r_N^k(1,p,\xi) = |\xi|^{m-\frac{1}{2}} r_N^k(|\xi|, p, |\xi|^{-1}\xi), \tag{4.40}$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $N \in \mathbb{N} \cup 0$.

Note that at fixed $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, α, $\beta \in \mathbb{Z}_+^2$ function $D_p^\alpha D_\xi^\beta g(\cdot, p, \xi)$ belongs to $C_c^\infty(\mathbb{R})$. Hence, its Fourier transform belongs to the Schwartz space and there exists such constant $C_{4,k}^{\alpha,\beta} > 0$ that

$$\int_{-\infty}^{+\infty} |t|^{2(N+1)} \left| D_p^\alpha D_\xi^\beta \widehat{g}(t,p,|\xi|^{-1}\xi) \right| dt \le C_{4,k}^{\alpha,\beta} |\xi|^{-|\beta|}, \tag{4.41}$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $|\xi| \ge 1$.

It follows from formulas (4.32), (4.40), (4.41) that

$$r_N^k(1,p,\xi) \in \widetilde{S}^{m-N-\frac{3}{2}}(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k)), \quad k = 1,\ldots,N_2,\ N \in \mathbb{N} \cup 0. \tag{4.42}$$

Step 6: final part of the proof. Using formula (4.17), taking into account that $f(\cdot,\xi)$ has the compact support at fixed $\xi \in \Sigma$ and integrating by parts in (4.19) we obtain

$$I_2^k(p,\xi) \in \widetilde{S}^{-\infty}(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k)), \quad k = 1,\ldots,N_2. \tag{4.43}$$

Formulas (4.18), (4.28), (4.43) imply that

$$e^{-iS^*(p,\xi)} I(p,\xi) \in \widetilde{S}^{m-\frac{1}{2}}(\widetilde{P}(\widetilde{p}^k) \times \widetilde{\Sigma}(\widetilde{\xi}^k)), \quad k = 1,\ldots,N_2. \tag{4.44}$$

Finally, using (4.13), (4.44) we get

$$e^{-iS^*(p,\xi)} I(p,\xi) \in S^{m-\frac{1}{2}}(P_{2\epsilon} \times \Sigma_{2\epsilon}).$$

Since we can chose ϵ arbitrarily small, we obtain (4.3).

Denote

$$r_N(p,\xi) = e^{-iS^*(p,\xi)} I(p,\xi) - \sum\nolimits_{j=0}^{N} a_j(p,\xi),$$

where $p \in P$, $\xi \in \Sigma$, $N \in \mathbb{N} \cup 0$. It follows from formulas (4.18), (4.38) that

$$r_N(p,\xi) = e^{-iS^*(p,\xi)} I_2^k(p,\xi) + (2\pi)^{-\frac{1}{2}} e^{i\frac{\pi}{4}} r_N^k(1,p,\xi), \tag{4.45}$$

where $p \in \widetilde{P}(\widetilde{p}^k)$, $\xi \in \widetilde{\Sigma}(\widetilde{\xi}^k)$, $k = 1, \ldots, N_2$ and $N \in \mathbb{N} \cup 0$.

Using formulas (4.13), (4.42), (4.43), (4.45) we obtain

$$r_N(p,\xi) \in S^{m-N-\frac{3}{2}}(P_{2\epsilon} \times \Sigma_{2\epsilon}).$$

Since ϵ can be chosen arbitrarily small, we get (4.6).

Uniqueness of functions a_j, $j \in \mathbb{N} \cup 0$, follows from uniqueness of asymptotic expansion (4.6) of function $e^{-iS^*(p,\xi)} I(p,\xi)$. Lemma 4.1 is proved. $\square$

5 Proof of Lemma 3.1

In this section we prove Lemma 3.1. The proof is divided into five lemmas.

Lemma 5.1. *Let $\alpha \in \mathbb{R} \setminus \{0,1\}$ and $0 < \delta < \frac{1}{2}$ be fixed. Then $R^*_{\alpha,\delta} R_{\alpha,\delta}$ is a proper pseudo-differential operator.*

Proof. Note that the following statements are true:

1. C_α doesn't contain points of types $(p, 0, x, \xi dx)$, $(p, \eta dp, x, 0)$ and C_α^t doesn't contain points of types $(x, 0, p, \eta dp)$, $(x, \xi dx, p, 0)$;
2. the support of $K_{\alpha,\delta}$ (resp. $K^*_{\alpha,\delta}$) is compact in $P \times X$ (resp. $X \times P$), where $K_{\alpha,\delta}$ is the Schwartz kernel of $R_{\alpha,\delta}$ defined in (3.6) and $K^*_{\alpha,\delta}$ is the Schwartz kernel of $R^*_{\alpha,\delta}$, given by the same formula (3.6) after changing p to x and vice versa;
3. submanifolds $C_\alpha^t \times C_\alpha$ and $T^*X \times \mathrm{diag}(T^*P \times T^*P) \times T^*X$ (where diag is defined in (3.37)) of $T^*X \times T^*P \times T^*P \times T^*X$ intersect transversally.

Statement 3 can be proved in the following way. Note that

$$\begin{aligned} &\big(C_\alpha^t \times C_\alpha\big) \cap \big(T^*X \times \mathrm{diag}(T^*P \times T^*P) \times T^*X\big) = \\ &= \Big\{(x, \lambda \sum\nolimits_{j=1}^{2} p_j^\alpha x_j^{\alpha-1} dx_j, p, -\lambda \sum\nolimits_{j=1}^{2} x_j^\alpha p_j^{\alpha-1} dp_j, p, -\lambda \sum\nolimits_{j=1}^{2} x_j^\alpha p_j^{\alpha-1} dp_j, \\ &\quad x, \lambda \sum\nolimits_{j=1}^{2} p_j^\alpha x_j^{\alpha-1} dx_j) \colon x \in X,\ p \in P,\ q_\alpha(p_1x_1, p_2x_2) = 1,\ \lambda \in \mathbb{R} \setminus 0\Big\}. \end{aligned}$$

It follows from this formula and from formulas (2.2), (2.4), (3.11) that

$$\begin{gathered} \dim\big(C_\alpha^t \times C_\alpha\big) = 8,\ \ \dim\big(T^*X \times \mathrm{diag}(T^*P \times T^*P) \times T^*X\big) = 12, \\ \dim\big((C_\alpha^t \times C_\alpha) \cap (T^*X \times \mathrm{diag}(T^*P \times T^*P) \times T^*X)\big) = 4. \end{gathered}$$

Hence, the dimension of sum of tangent spaces of $C_\alpha^t \times C_\alpha$ and of $T^*X \times \operatorname{diag}(T^*P \times T^*P) \times T^*X$ in points of interection of these submanifolds is equal to dimension of $T^*X \times T^*P \times T^*P \times T^*X$. Statement 3 is proved.

It follows from statements 1, 2, 3 above that $R_{\alpha,\delta}^* R_{\alpha,\delta}$ is a Fourier integral operator associated with canonical relation $C_\alpha^t \circ C_\alpha \subset \operatorname{diag}(T^*X \times T^*X)$ (see (3.38) and Ref. [Du], Theorem 2.4.1). Hence, $R_{\alpha,\delta}^* R_{\alpha,\delta}$ is a pseudo-differential operator.

Operator $R_{\alpha,\delta}^* R_{\alpha,\delta}$ is proper since operators $R_{\alpha,\delta}^*$ and $R_{\alpha,\delta}$ have compactly supported Schwartz kernels. $\square$

The remaining part of this subsection is devoted to computation of principal symbol of $R_{\alpha,\delta}^* R_{\alpha,\delta}$.

We need to introduce some notations. Note that at fixed $p = (p_1, p_2) \in P$ the curve $X_{\alpha,p}$ is the graph of function

$$\begin{gathered} x_2 = x_2(x_1, p) \stackrel{\text{def}}{=\!=} p_2^{-1}(1 - p_1^\alpha x_1^\alpha)^{1/\alpha}, \\ 0 < x_1 < p_1^{-1}, \quad \text{if } \alpha > 0, \\ p_1^{-1} < x_1 < +\infty, \quad \text{if } \alpha < 0. \end{gathered}$$

Similarly, at fixed $x = (x_1, x_2) \in X$ the curve $P_{\alpha,x}$ is the graph of function

$$\begin{gathered} p_2 = p_2(p_1, x) \stackrel{\text{def}}{=\!=} x_2^{-1}(1 - p_1^\alpha x_1^\alpha)^{1/\alpha}, \\ 0 < p_1 < x_1^{-1}, \quad \text{if } \alpha > 0, \\ x_1^{-1} < p_1 < +\infty, \quad \text{if } \alpha < 0. \end{gathered}$$

Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and $\xi = (\xi_1, \xi_2) \in \Sigma \cup (-\Sigma)$ be fixed, $1/\alpha + 1/\beta = 1$. Define

$$S(x, \xi) = \xi_1 x_1 + \xi_2 x_2, \quad x = (x_1, x_2) \in X, \tag{5.1}$$

$$U(p, \xi) = (\operatorname{sgn} \xi_1)\big(|\xi_1|^\beta p_1^{-\beta} + |\xi_2|^\beta p_2^{-\beta}\big)^{1/\beta}, \quad p = (p_1, p_2) \in P. \tag{5.2}$$

Restricting functions $S(x, \xi)$ and $U(p, \xi)$ to $X_{\alpha,p}$ and $P_{\alpha,x}$, respectively, we obtain the following functions:

$$\begin{aligned} \widetilde{S}(x_1, p, \xi) &\stackrel{\text{def}}{=\!=} S((x_1, x_2(x_1, p)), \xi) \\ &= \xi_1 x_1 + \xi_2 p_2^{-1}(1 - p_1^\alpha x_1^\alpha)^{1/\alpha}, \\ \widetilde{U}(p_1, x, \xi) &\stackrel{\text{def}}{=\!=} U((p_1, p_2(p_1, x)), \xi). \end{aligned}$$

Lemma 5.2. *Let $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and $\xi = (\xi_1, \xi_2) \in \Sigma \cup (-\Sigma)$ be fixed. Let $1/\alpha + 1/\beta = 1$. Then:*

1. *at fixed $p = (p_1, p_2) \in P$ function $S(\cdot, \xi)$ has the unique local extremum $(x_1^*(p, \xi), x_2^*(p, \xi))$ on $X_{\alpha,p}$ and the following formulas are valid:*

$$x_j^*(p, \xi) = |U(p, \xi)|^{1-\beta} |\xi_j|^{\beta-1} p_j^{-\beta}, \quad j = 1, 2, \tag{5.3}$$

$$S\big((x_1^*(p, \xi), x_2^*(p, \xi)), \xi\big) = U(x, \xi), \tag{5.4}$$

$$\widetilde{S}_{x_1 x_1}\big(x_1^*(p, \xi), p, \xi\big) = (1 - \alpha)|U(p, \xi)|^{2\beta - 1} (p_1 p_2)^\beta \xi_1 |\xi_1|^{1-\beta} |\xi_2|^{-\beta}; \tag{5.5}$$

2. *at fixed* $x = (x_1, x_2) \in X$ *function* $U(\cdot, \xi)$ *has the unique local extremum on* $P_{\alpha,x}$ *given by formula* (3.10) *and the following formulas are valid:*

$$U\big((p_1^*(x,\xi), p_2^*(x,\xi)), \xi\big) = S(x,\xi), \tag{5.6}$$

$$\widetilde{U}_{p_1p_1}\big(p_1^*(x,\xi), x, \xi\big) = \alpha\beta |S(x,\xi)|^{1+\frac{2}{\alpha}} x_1^{1+\frac{2}{\beta}} x_2^{-1} |\xi_1|^{1-\frac{2}{\alpha}} \xi_2^{-1}. \tag{5.7}$$

Proof. 1. Derivatives of function $\widetilde{S}(\cdot, p, \xi)$ are given by the following formulas:

$$\begin{aligned} \widetilde{S}_{x_1}(x_1, p, \xi) &= \xi_1 - \xi_2 p_1^{\alpha} p_2^{-1} x_1^{\alpha-1}\big(1 - p_1^{\alpha} x_1^{\alpha}\big)^{\frac{1}{\alpha}-1}, \\ \widetilde{S}_{x_1x_1}(x_1, p, \xi) &= (1-\alpha) p_2^{-1} \xi_2 \big(1 - p_1^{\alpha} x_1^{\alpha}\big)^{\frac{1}{\alpha}-2} p_1^{\alpha} x_1^{\alpha-2}. \end{aligned} \tag{5.8}$$

Solving equation $\widetilde{S}_{x_1}(x_1, p, \xi) = 0$ at fixed $p \in P$, $\xi \in \Sigma \cup (-\Sigma)$ for x_1, we find the unique point $x_1^*(p,\xi)$ given by (5.3).

Formula (5.4) follows from (5.3) and formula (5.5) follows from (5.3), (5.8).

2. Derivatives of function $\widetilde{U}(\cdot, x, \xi)$ are given by the following formulas:

$$\begin{aligned} \widetilde{U}_{p_1}(p_1, x, \xi) &= -(\operatorname{sgn}\xi_1)\, p_1^{\alpha-1} |\widetilde{U}(p_1, x, \xi)|^{1-\beta} \times \\ &\quad \times \big(|\xi_1|^{\beta} p_1^{-\alpha\beta} - |\xi_2|^{\beta} x_1^{\alpha} x_2^{\beta} (1 - p_1^{\alpha} x_1^{\alpha})^{-\beta}\big), \\ \widetilde{U}_{p_1p_1}(p_1, x, \xi) &= (1-\beta)\big(\widetilde{U}(p_1, x, \xi)\big)^{-1} \big(\widetilde{U}_{p_1}(p_1, x, \xi)\big)^2 + \\ &\quad + \alpha\beta (\operatorname{sgn}\xi_1)\, p_1^{\alpha-2} |\widetilde{U}(p_1, x, \xi)|^{1-\beta} \times \\ &\quad \times \Big(|\xi_1|^{\beta} p_1^{-\alpha\beta} + |\xi_2|^{\beta} p_2^{-\alpha(1+\beta)} x_1^{2\alpha} x_2^{-2\alpha} p_1^{\alpha}\Big). \end{aligned} \tag{5.9}$$

Solving equation $\widetilde{U}_{p_1}(p_1, x, \xi) = 0$ at fixed $x \in X$, $\xi \in \Sigma \cup (-\Sigma)$ for p_1 we obtain the unique point $p_1^*(x,\xi)$ given by (3.10).

Formula (5.6) follows from (3.10) and formula (5.7) follows from formulas (3.10), (5.9). $\square$

Lemma 5.3. *Let* $\alpha \in \mathbb{R} \setminus \{0, 1\}$ *and* $0 < \delta < \frac{1}{2}$ *be fixed. Then:*

1. *there exists* $\delta' = \delta'(\alpha, \delta) > 0$ *such that for any* $u \in C^{\infty}(X)$

$$(R_{\alpha,\delta} e^{iS(\cdot,\xi)} u)(p) \in S^{-\infty}\big(P \times \big(\mathbb{R}^2 \setminus (0 \cup \Sigma_{\delta'} \cup -\Sigma_{\delta'})\big)\big),$$

where S *is defined in* (5.1), $\Sigma_{\delta'}$ *is defined in* (3.8)*;*

2. *there exists* $\delta'' = \delta''(\alpha, \delta) > 0$ *such that for any* $v \in C^{\infty}(P)$

$$(R^*_{\alpha,\delta} e^{iU(\cdot,\xi)} v)(x) \in S^{-\infty}\big(X \times \big(\mathbb{R}^2 \setminus (0 \cup \Sigma_{\delta''} \cup -\Sigma_{\delta''})\big)\big),$$

where U *is defined in* (5.2), $\Sigma_{\delta''}$ *is defined in* (3.8).

Proof. 1. It follows from (3.5) that it is sufficient to show that there exists $\delta' = \delta'(\alpha, \delta) > 0$ such that function $S(\cdot, \xi)$ defined in (5.1) at fixed $\xi \in \mathbb{R}^2 \setminus (0 \cup \Sigma_{\delta'} \cup -\Sigma_{\delta'})$ doesn't have local extrema on $X_{\alpha,p} \cap \operatorname{supp} æ_{\alpha,\delta}$ for all $p \in P$.

Note that $S(\cdot,\xi)$ at fixed $\xi \in \mathbb{R}^2 \setminus (0 \cup \Sigma \cup -\Sigma)$ doesn't have local extrema on curve $X_{\alpha,p}$ for all $p \in P$.

It follows from definition (3.3) that

$$æ_{\alpha,\delta}(p,x) = 0, \ \text{ if } p \notin P_\delta \text{ or } x \notin X_\delta, \tag{5.10}$$

where X_δ, P_δ are defined in formula (3.14).

Using Lemma 5.2 (1) we obtain that at fixed $p \in P$, $\xi \in \Sigma \cup (-\Sigma)$ function $S(\cdot,\xi)$ has the unique local extremum $(x_1^*(p,\xi), x_2^*(p,\xi))$ on $X_{\alpha,p}$ given by formula (5.3). It follows from formula (5.3) that there exists such $\delta' = \delta'(\alpha,\delta) > 0$ that for any $p \in P_{\delta/2}$, $\xi = (\xi_1,\xi_2) \in \Sigma \cup (-\Sigma)$ such that $\xi_1/\xi_2 \le \delta'$ or $\xi_2/\xi_1 \le \delta'$, we have $x^*(p,\xi) \notin X_{\delta/2}$.

Hence, using (5.10) we obtain that if $p \in P$, $\xi \in (\Sigma \setminus \Sigma_\delta) \cup ((-\Sigma) \setminus (-\Sigma_\delta))$, then $x^*(p,\xi) \notin X_{\alpha,p} \cap \operatorname{supp} æ_{\alpha,\delta}$. The first statement of Lemma 5.3 is proved.

2. The second statement can be proved in a similar way. □

Lemma 5.4. *Let $\alpha \in \mathbb{R} \setminus \{0,1\}$ and $0 < \delta < \frac{1}{2}$ be fixed. Let $1/\alpha + 1/\beta = 1$. Then there exist the unique functions $a_j \in C^\infty(P \times \mathbb{R}^2 \setminus 0)$, $j \in \mathbb{N} \cup 0$, satisfying*

$$a_j(p,\lambda\xi) = \lambda^{-j-\frac{1}{2}} a_j(p,\xi), \quad p \in P, \ \xi \in \mathbb{R}^2 \setminus 0, \ \lambda > 0, \tag{5.11}$$

and such that the following formula holds:

$$e^{-iU(p,\xi)}(R_{\alpha,\delta} e^{iS(\cdot,\xi)})(p) - \sum\nolimits_{j=0}^{N} a_j(p,\xi) \in S^{-N-\frac{3}{2}}(P \times \mathbb{R}^2 \setminus 0), \tag{5.12}$$

where $N \in \mathbb{N} \cup 0$, S is defined in (5.1), U is defined in (5.2).

Furthermore, the following formula holds:

$$\begin{aligned} a_0(p,\xi) = (2\pi)^{\frac{1}{2}} |1-\alpha|^{-\frac{1}{2}} \exp\big(i(\pi/4)\cdot \operatorname{sgn}(1-\alpha)\big) \times \\ \times (p_1p_2)^{-\frac{\beta}{2}} |\xi_1\xi_2|^{\frac{\beta}{2}-1} |U(p,\xi)|^{\frac{3}{2}-\beta} æ_{\alpha,\delta}\big(p, (x_1^*(p,\xi), x_2^*(p,\xi))\big), \end{aligned} \tag{5.13}$$

where $p \in P$, $\xi \in \Sigma \cup (-\Sigma)$; $x_1^*(p,\xi)$, $x_2^*(p,\xi)$ are defined in (5.3).

Proof. We will prove this lemma using Lemma 4.1 of Section 4.

Note that function q_α satisfies assumptions Q1, Q2 of Remark 2.2 and function S defined in (5.1) satisfies assumption S1 of Section 4 with $\widehat{\Sigma} = \Sigma$ and $\widehat{\Sigma} = -\Sigma$. It follows from Lemma 5.2 (1) that functions q_α, S satisfy assumptions QS1, QS2 of Section 4.

Using formulas (5.3), (5.5) of Lemma 5.2 we also obtain the following equality:

$$\begin{aligned} |\widetilde{S}_{x_1x_1}(x_1^*(p,\xi),p,\xi)|^{\frac{1}{2}} \partial_2 q_\alpha\big(p_1x_1^*(p,\xi), p_2x_2^*(p,\xi)\big) = \\ = |1-\alpha|^{\frac{1}{2}} (p_1p_2)^{\frac{\beta}{2}} |\xi_1\xi_2|^{1-\frac{\beta}{2}} |U(p,\xi)|^{\beta-\frac{3}{2}}, \end{aligned} \tag{5.14}$$

where $p \in P$, $\xi \in \Sigma \cup (-\Sigma)$.

Now Lemma 5.4 follows from Lemmas 4.1, 5.3 (1) if we take into account that asymptotic expansions are unique. □

The following lemma finishes the proof of Lemma 3.1.

Lemma 5.5. *Let $\alpha \in \mathbb{R} \setminus \{0,1\}$ and $0 < \delta < \frac{1}{2}$ be fixed. Let $1/\alpha + 1/\beta = 1$. Then there exist the unique functions $b_m \in C^\infty(X \times \mathbb{R}^2 \setminus 0)$, $m \in \mathbb{N} \cup 0$, satisfying*

$$b_m(x, \lambda\xi) = \lambda^{-m-1} b_m(x,\xi), \quad x \in X,\ \xi \in \mathbb{R}^2 \setminus 0,\ \lambda > 0, \tag{5.15}$$

and such that the following formula is valid:

$$e^{-i\xi x}(R^*_{\alpha,\delta} R_{\alpha,\delta} e^{iS(\cdot,\xi)})(x) - \sum\nolimits_{m=0}^{N} b_m(x,\xi) \in S^{-N-2}(X \times \mathbb{R}^2 \setminus 0), \tag{5.16}$$

where $N \in \mathbb{N} \cup 0$, S is defined in (5.1).

Furthermore,

$$\begin{aligned} b_0(x,\xi) &= 2\pi|\alpha|^{-1} \exp\bigl(i(\pi/4)\cdot \operatorname{sgn}(1-\alpha)\cdot(1-\operatorname{sgn}\xi_1)\bigr)\times \\ &\times (x_1x_2)^{-\frac{1}{\beta}} |\xi_1\xi_2|^{-\frac{1}{\beta}} |\xi x|^{1-\frac{2}{\alpha}} æ^2_{\alpha,\delta}\bigl((p_1^*(x,\xi), p_2^*(x,\xi)), x\bigr), \end{aligned} \tag{5.17}$$

where $x \in X$, $\xi \in \Sigma \cup (-\Sigma)$; $p_1^*(x,\xi)$, $p_2^*(x,\xi)$ are defined in (3.10).

Besides, there exists $\epsilon = \epsilon(\alpha,\delta) > 0$ such that $b_m(x,\xi) = 0$ for $x \in X$, $\xi \in \mathbb{R}^2 \setminus (0 \cup \Sigma_\epsilon \cup -\Sigma_\epsilon)$, $m \geq 0$, where Σ_ϵ is defined by formula (3.8).

Proof. Using Lemma 5.4 we obtain the expansion:

$$e^{-iU(p,\xi)}(R_{\alpha,\delta} e^{iS(\cdot,\xi)})(p) = \sum\nolimits_{j=0}^{N} a_j(p,\xi) + r_N(p,\xi), \tag{5.18}$$

where $p \in P$, $\xi \in \mathbb{R}^2 \setminus 0$, $r_N \in S^{-N-\frac{3}{2}}(P \times \mathbb{R}^2 \setminus 0)$, $N \in \mathbb{N} \cup 0$.

Note that function q_α satisfies assumptions Q1, Q2 of Remark 2.2 and function U defined in (5.2) satisfies assumption S1 of Section 4 with $\widehat{\Sigma} = \Sigma$ and $\widehat{\widehat{\Sigma}} = -\Sigma$. It follows from Lemma 5.2 (2) that functions q_α, U also satisfy assumptions QS1, QS2 of Section 4.

Using formulas (3.10), (5.7) we also obtain the following formula:

$$\begin{aligned} &|\widetilde{U}_{p_1p_1}(p_1^*(x,\xi), x, \xi)|^{\frac{1}{2}} \partial_2 q_\alpha\bigl(p_1^*(x,\xi)x_1, p_2^*(x,\xi)x_2\bigr) = \\ &= |\alpha\beta|^{\frac{1}{2}} (x_1x_2)^{\frac{1}{2}+\frac{1}{\beta}} |\xi_1\xi_2|^{\frac{1}{2}-\frac{1}{\alpha}} |\xi x|^{\frac{2}{\alpha}-\frac{1}{2}}, \end{aligned} \tag{5.19}$$

where $x \in X$, $\xi \in \Sigma \cup (-\Sigma)$; $p_1^*(x,\xi)$, $p_2^*(x,\xi)$ are defined in (3.10).

Using Lemmas 4.1, 5.3 (2) we obtain that there exist the unique functions $b_{j,k} \in C^\infty(X \times \mathbb{R}^2 \setminus 0)$, j, $k \in \mathbb{N} \cup 0$, satisfying

$$b_{j,k}(x, \lambda\xi) = \lambda^{-j-k-1} b_{j,k}(x,\xi), \quad x \in X,\ \xi \in \mathbb{R}^2 \setminus 0,\ \lambda > 0, \tag{5.20}$$

and such that the following formula holds:

$$e^{-i\xi x} R^*_{\alpha,\delta}\left(e^{iU(\cdot,\xi)} a_j(\cdot,\xi)\right)(x) - \sum_{k=0}^{M} b_{j,k}(x,\xi) \in S^{-j-M-2}\bigl(X \times \mathbb{R}^2 \setminus 0\bigr), \tag{5.21}$$

where $j,\ M \in \mathbb{N} \cup 0$. Furthermore, it follows from Lemma 5.3 (2) that there exists $\epsilon = \delta''(\alpha, \delta) > 0$ such that $b_{j,k}(x, \xi) = 0$ if $x \in X,\ \xi \in \mathbb{R}^d \setminus (0 \cup \Sigma_\epsilon \cup -\Sigma_\epsilon)$ for all $j,\ k \in \mathbb{N} \cup 0$.

Besides, denoting $b_0 = b_{0,0}$ we obtain (5.17).

Taking into account that $r_N \in S^{-N-\frac{3}{2}}(P \times \mathbb{R}^2 \setminus 0)$ and using Lemmas 4.1, 5.3 (2), we obtain that

$$e^{-i\xi x} R^*_{\alpha,\delta} \left(e^{iU(\cdot,\xi)} r_N(\cdot, \xi) \right)(x) \in S^{-N-2}\big(X \times \mathbb{R}^2 \setminus 0\big). \tag{5.22}$$

Denote $b_m = \sum_{j+k=m} b_{j,k},\ m \in \mathbb{N}$. Now Lemma 5.5 follows from (5.18), (5.20), (5.21), (5.22) if we take into account that asymptotic expansions are unique. $\square$

Lemmas 5.1, 5.5 imply the statement of Lemma 3.1 if we take into account that the full symbol of $R^*_{\alpha,\delta} R_{\alpha,\delta}$ is given by formula

$$\sigma_{\alpha,\delta}(x, \xi dx) = e^{-i\xi x}(R^*_{\alpha,\delta} R_{\alpha,\delta} e^{iS(\cdot,\xi)})(x).$$

and the principal symbol $\sigma^P_{\alpha,\delta}(x, \xi dx)$ is the term of top degree in the asymptotic expansion of $\sigma_{\alpha,\delta}(x, \xi dx)$.

6 Numerical examples

In the work [Ag2] an explicit formula for inversion of transform $R_\alpha,\ \alpha \in \mathbb{R} \setminus 0$, on the space $C^3_c(X)$ of smooth functions with compact support was obtained. But Theorem 2.1 shows that the transform R_α is defined on a much more general space $\mathscr{E}'(X)$ of distributions with compact support. Proposition 2.1 together with Theorem 2.1 (3) show that in this case it is at least possible to reconstruct the singularities $(x, \xi dx) \in T^*X$ of a distribution $u \in \mathscr{E}'(X)$ such that $\xi = (\xi_1, \xi_2),\ \xi_1\xi_2 > 0$. We will illustrate this by several examples.

Also note that this reconstruction of singularities is stable with respect to perturbations of class $C^\infty_c(X)$ since the Fourier integral operators map smooth functions to smooth functions.

1. Visibility of singularities. Let u be the characteristic function of the black figure at Fig. 1 (a). Note that the wavefront set of u consists of such pairs $(x, \xi dx)$ that x belongs to the boundary point of the black figure and $\xi \neq 0$ belongs to the linear hull of the normal cone to the black figure at x.

Consider the transform R_α with $\alpha = 0.5$ and chose $\varepsilon = 0.49$. In notations of Proposition 2.1 we have $\gamma(\alpha) = 5,\ \delta > 0.003$. Note that for any $x = (x_1, x_2) \in \operatorname{supp} u$ we have $x_1,\ x_2 \in (\varepsilon, \varepsilon^{-1})$. Besides, for any $(x, \xi dx) \in WF(u)$ such that x is not a corner of the black figure we have that $\varepsilon < \frac{\xi_1}{\xi_2} < \varepsilon^{-1}$. Set $\delta^* = 0.003$. Applying Theorem 2.1 (3) and Proposition 2.1 we obtain that $u^* = R^*_\alpha \chi^1_{\delta^*} \chi^2_{\delta^*} R_\alpha u$ has the same singularities $(x, \xi dx)$ as function u everywhere except the corner points of the black figure.

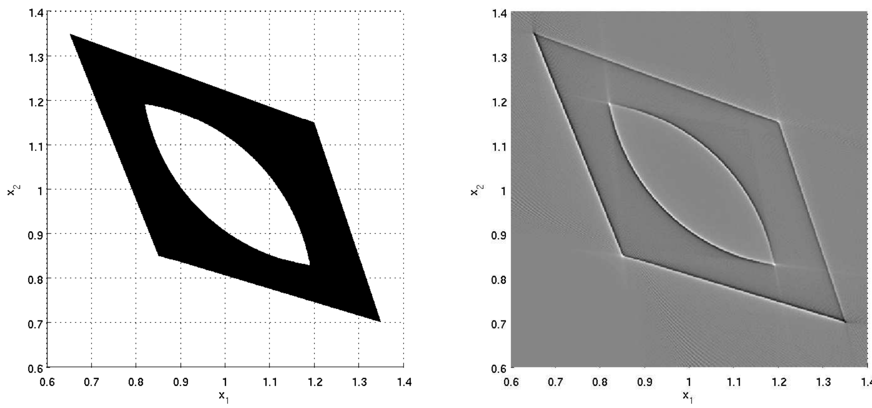


Figure 1: visibility: (a) original figure; (b) reconstructed function.

The function u^* is presented in Fig. 1 (b). We see that the numerical result is consistent with the statements of Theorem 2.1 (3) and Proposition 2.1 so that the singularities of u^* are precisely the singularities of u.

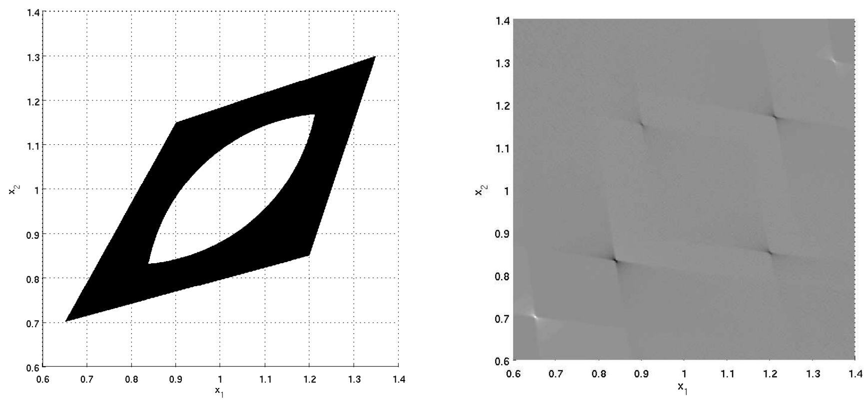
2. Invisibility of singularities. Now we consider the worst possible case. Note that all the singularities of function $u^* = R^*_\alpha \chi^1_{\delta^*}\chi^2_{\delta^*} R_\alpha u$ are contained in the singularities of function u so that the worst possible case arises when u has singularities but u^* doesn't.

Consider, as above, the transform R_α with $\alpha = 0.5$ and set $\varepsilon = 0.49$ so that $\gamma(\alpha) = 5$, $\delta > 0.003$. Set $\delta^* = 0.003$. Let u be the characteristic function of the black figure at Fig. 2 (a) so that for any $x = (x_1, x_2) \in \operatorname{supp} u$ we have $x_1,\ x_2 \in (\varepsilon, \varepsilon^{-1})$ and for any $(x, \xi dx) \in WF(u)$ we have that $\frac{\xi_1}{\xi_2} < 0$, where $\xi = (\xi_1, \xi_2)$ and x is not a corner point of the black figure. Then, according to Theorem 2.1 (3) and Proposition 2.1, the function u^* can have singularities only in corner points of the black figure.

The function u^* is presented in Fig. 2 (b). Once again, the numerical reconstruction is consistent with the statements of Theorem 2.1 (3) and Proposition 2.1 since u^* doesn't have singularities but in the corner points of the black figure.

3. Partial visibility. Let u be the characteristic function of the black figure at Fig. 3 (a). In this case the wavefront set of u consists of the pairs $(x, \xi dx)$ where x belongs to the boundary of the black figure and $\xi \neq 0$ is orthogonal to the boundary at point x.

We chose $\alpha = 0.5$, $\gamma(\alpha) = 5$, $\varepsilon = 0.25$ so that $\delta > 10^{-4}$. Take $\delta^* = 10^{-4}$ and set $u^* = R^*_\alpha \chi^1_{\delta^*}\chi^2_{\delta^*} R_\alpha u$. According to Theorem 2.1 (3) and Proposition 2.1 function u^* must have singularities at points $(x, \xi dx)$ where x belongs to the boundary of the black figure, $\xi \neq 0$ is orthogonal to the boundary of the black figure at point x and $14^\circ \approx \arctan \varepsilon < \arctan(\frac{\xi_1}{\xi_2}) < \arctan \varepsilon^{-1} \approx 76^\circ$. Also, according to Theorem 2.1 (3) u^* mustn't have singularities if either x doesn't



(a) (b)

Figure 2: insivibility: (a) original function; (b) reconstructed function.

belong to the boundary of the black figure or $\xi_1\xi_2 < 0$.

Function u^* is presented at Fig. 3 (b). We see that the result of numerical simulation confirms the statements of Theorem 2.1 (3) and Proposition 2.1.

7 Aknowledgements

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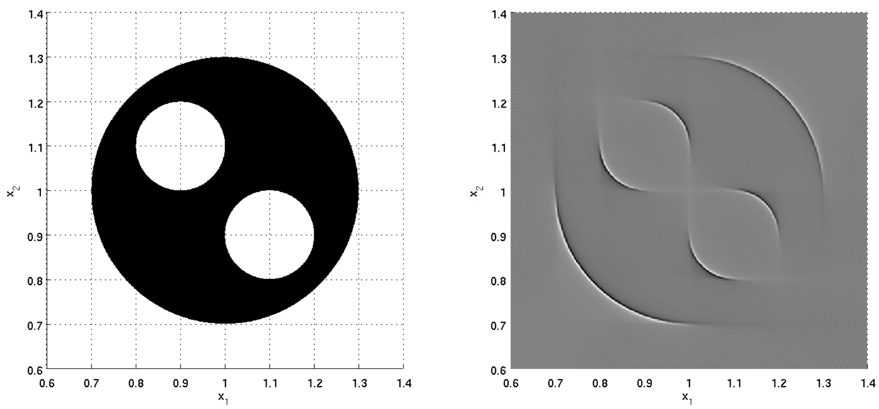

(a) (b)

Figure 3: partial visibility: (a) original function; (b) reconstructed function.

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